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A Model of Industrialization and Rural Income Distribution

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February 4, 2019

Abstract

We develop a simple dynamic general equilibrium model to analytically characterize how (de) industrialization interacts with rural income distribution and also explore the implications for aggregate GDP growth, the evolution of rural income distribution as well as welfare. The model features a novel non-homothetic utility function that greatly improves model tractability. Redistributive policies are shown to sometimes enhance GDP and welfare by (1) boosting the production of the goods with high desirability (or productivity) but constrained by depressed demand due to income inequality, and (2) internalizing the dynamic impact of private production and consumption decisions on future public productivities.

Key Words: Structural Change, Income Distribution, Non-homothetic Preference, Human Capital, Economic Growth

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Introduction

Less developed countries are all featured by unfinished industrialization and a large proportion of rural population. How labor is reallocated from the agriculture sector to the non-agriculture sector (industrialization) and how rural income distribution evolves over time are two important structural processes of economic development. The primary objective of this paper is to explore how these two processes interact with each other and their implications for economic growth and welfare. Whereas the existing pertinent literature studies these issues mainly from the partial-equilibrium and empirical perspectives, we will take a general-equilibrium and theoretical approach. The advantages of this different approach are obvious. First, any market forces that drive industrialization and income distribution must involve changes in prices of output and production factors, which should be endogenously

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explained rather than taken as exogenous as in all partial-equilibrium analyses. Second, we are still lack of sufficient understanding about the theoretical mechanisms how industrialization (structural change) and evolution of rural income distributions take place and interact with each other.

Therefore, we develop a heterogeneous-agent dynamic general equilibrium model with two sectors: agriculture and non-agriculture (including manufacturing and service). Households have non-homothetic preferences over agriculture and non-agriculture goods following Engle's law, which also serves as one of the important mechanisms that drive structural change. Moreover, households are heterogeneous in their human capital endowment, which is the root cause of income inequality. Due to the non-homothetic preference, micro-level income heterogeneity has a macro impact on the aggregate economy in terms of GDP level and its growth rate, sectorial reallocation of production resources, and rural/urban income distribution. The economic dynamics are driven by the sectorial productivity growth, which is in turn endogenous to the human capital allocation across sectors. We analytically characterize how initial sectorial productivities and household heterogeneity in human capital endowment jointly determine the levels and dynamics of employment shares, value-added shares, productivities, Gini coefficients of different sectors and the GDP growth rate, both on the transitional dynamics and in the long-run steady state.

To facilitate our understanding of the mechanisms, we divide the model into two parts: static and dynamic. In the static part (Section 2), sectorial productivities are exogenous and they fully determine prices and incomes, which in turn determine demand and supply in both sectors, and hence resource allocation across sectors, rural and urban income distribution, as well as the aggregate output. We examine three different possible scenarios, namely, all households consume both agriculture and non-agriculture products, only rich households consume both products, and no households consume non-agriculture products. They translate into three different economic structures, manifested as endogenously different functional forms of the aggregate production function, which is a common technical feature of models in New Structural Economics (see Ju, et al. 2015, Lin and Wang 2018). Moreover, we show that, changes in relative sectorial productivities that result in the advance of industrialization may sometimes lead to non-monotonic changes in rural income inequality, depending on the extent to which rich and poor households are heterogeneous in their human capital endowment and their proportions in the population. Notice that value-added shares and employment shares in each sector are not necessarily equal in our model because workers are heterogeneous in human capital endowment.

In the dynamic part (Section 3), sectorial productivities are endogenously changing, depending on the human capital allocation across the two sectors, which in turn depends on demand and supply of products in the two sectors governed by sectorial productivities and income distributions. A key feature of the dynamic equilibrium is path dependence: different levels of the initial sector productivities may lead to diametrically opposite processes of structural change and polarized steady states in the long run. More concretely, we show that there exists a unique steady state, in which the two sectors grow at the same constant rate without structural change and both rural and urban Gini coefficients stay unchanged. Moreover, the value-added share of the non-agriculture sector is independent of household heterogeneity or aggregate factor endowment, and it strictly increases with the price demand elasticity of

agriculture goods and how strong the learning externality is in human capital. The aggregate GDP growth rate strictly increases with price demand elasticity of agriculture goods and the aggregate human capital endowment, but is independent of household heterogeneity. However, the sectorial employment shares and rural Gini coefficient do depend on the details of household heterogeneity. We also show that this steady state is unstable. Any small deviation from the steady state results in permanent divergence away from it, leading either to continuous industrialization that ultimately converges to an asymptotic steady state without agriculture, or to continuous de-industrialization till the economy reaches a new steady state with only agriculture. The rural Gini coefficient may change non-monotonically on the transitional dynamics, depending on the initial productivities and per household income ratio between rich and poor.
However, the Laissez-faire market equilibrium allocation in the static economy may be neither Pareto efficient nor GDP maximizing because marginal rate of substitution between agriculture and non-agriculture consumption (or equivalently, marginal rate of transformation between agriculture and non-agriculture inputs) may not be equalized across households with different income levels and non-homothetic preferences. We show how certain income redistribution policies could enhance total GDP. The dynamic market equilibrium is not Pareto efficient for an additional reason: human capital externality, that is, households' private decisions on which sector to work does not internalize the impact of their decisions on future productivities, similar to Lucas (2004). We show with several simple examples that the welfare-maximizing policies are not necessarily those which ensure the highest GDP growth rates; both initial productivities and details of household heterogeneity matter. All these policy analyses are in Section 4.

Our paper contributes to the literature of industrialization and structural change at large in several aspects. First, a key novel feature of our model is that the Engle's law is captured by a quasi-linear utility function, which differs from the standard non-homothetic functions in this literature. More specifically, whereas Stone-Geary utility function (see, for example, Kongasmut et al (2001)) and the sequentially satiated utility function with zero or one unit of consumption for each variety (see Mastuyama (2002) and Buera and Kaboski (2012) assume an exogenous level of minimum or maximum consumption of certain goods, our utility function does not make those restrictive assumptions. Our function also differs from the non-homothetic CES preference (Comin et al. (2018), Mastuyama (2018)) in that we impose constant price demand elasticity for agriculture goods but allow for variable income demand elasticity and variable substitution elasticity across sectors, but the opposite is true for the non-homothetic CES. Moreover, our utility function enormously helps improve the model tractability that brings new insights.[1] For example, we show that balanced sectorial growth (without structural change) is possible for both the long-run steady state and the transitional path with our utility function, whereas it is almost never possible with any other standard non-homothetic preferences in the pertinent literature. Given the fundamental importance of non-homothetic preferences in the literature of structural change this new utility function helps deepen our understanding on the mechanisms how non-homothetic preferences affect

[1] The analytical convenience of this quasi-linear utility function in the structural change models is also demonstrated when allowing for multiple production factors, non-competitive market structures, input-output linkages across sectors and international trade, see Li, Liu and Wang (2016) and Lin and Wang (2018).

industrialization and economic growth. [2] Second, our paper contributes to the literature of structural change by showing how (de)industrialization works when sectorial productivity changes are endogenous. Note that most existing models of industrialization treat changes in productivities as exogenous (see, for example, Retsuccia et al (2008), Henrrendorf et al (2014)), but we treat them as endogenous by following Lucas (2004) and Matsuyama (2002). It enables us to explore the dynamic impact of today's productivities and industrialization on future productivities and industrialization, resulting in strong path dependence.

Our paper also sheds light on the determination of rural income distribution and its evolution in the process of structural change and rural-urban migration. Instead of highlighting the role of migration barriers such as labor market frictions (see Harris and Todaro (1970), Restuccia et al (2008), Trevor and Zhu (2018)), financial market frictions (Lakagos, et al. 2014), we highlight the role of heterogeneous endowment in human capital in the structural change process, echoing the theme of New Structural Economics (Lin, 2011). Different from the human capital model in Lucas (2004), who assumes that rural and urban sectors produce the same good, we treat agriculture and non-agriculture as different goods both in terms of preference and technology. Murphy et al (1989) and Mastuyama (2002) study how income distribution affects industrialization with the presence of non-homothetic preferences, but not the reverse impact of industrialization on income distribution. Our paper examines both directions with a particular focus on rural income distribution. We show when and how the derived rural Gini coefficient may change non-monotonically with industrialization, depending on the initial productivities and details of human capital heterogeneity across households. Roles of redistributive policies are also discussed.

The rest of the paper is organized as follows. In Section 2 and Section 3, we develop a static model and a dynamic model of industrialization and income distribution, respectively, to characterize the decentralized Laissez-faire market equilibrium. Redistributive policies are discussed in Section 4. The last section concludes.

Static Model

Environment

Consider an economy populated by a continuum of households with measure equal to unity. Households can be divided into two groups: a rich group with total measure equal to $\theta \in (0,1)$ and a poor group with measure $1-\theta$. Households are identical within each group.

Preference All the households have the same instantaneous utility function $u(c)$, where final consumption c is given by

$$c = c_m + \frac{\epsilon}{\epsilon - 1} c_a^{\frac{\epsilon-1}{\epsilon}}, \ \epsilon > 1, \tag{1}$$

where c_m denotes the consumption of non-agriculture good m and c_a denotes consumption of agriculture good a. The parameter ϵ is the price elasticity of demand for good a, that is, consumption demand for good a increases by $\epsilon\%$ when its price decreases by 1%. We

[2] For more discussions on the role of non-homothetic preferences, see Bopport (2014) and Mastuyama and Ushchev (2018).

require that both c_m and c_a must be non-negative: $u(c) = -\infty$ if $c_m < 0$ or $c_a < 0$. $u(\cdot)$ is a strictly increasing and concave function. This quasi-linear preference captures the Engle's law: agriculture good a is a necessary good whereas non-agriculture goods m are more luxurious, that is, when income is sufficiently low, only good a is consumed; when income is sufficiently high, only demand for good m would increase.

Technology All the technologies are constant returns to scales. One unit of human capital (effective labor) produces A_m units of non-agriculture good. One unit of human capital produces A_a units of agriculture good a. That is,

$$F_m(L_m) = A_m L_m, \tag{2}$$

and

$$F_a(L_a) = A_a L_a. \tag{3}$$

Endowment and Market Structure Every household in this economy is endowed with one unit of time. Each household in the poor group is endowed with L_p units of human capital and each household in the rich group is endowed with L_r units of human capital. Assume $L_r > L_p > 0$. So rich people are more productive than poor people. All the markets are perfectly competitive.

Let W denote the wage rate per unit of human capital (effective labor). Then the income of a rich household (I_r) and that of a poor household (I_p) are respectively given by

$$I_r = WL_r; I_p = WL_p. \tag{4}$$

Obviously, $I_r > I_p$, which is the reason why we call them rich and poor, respectively.

Proposition 1. *The Gini coefficient for this economy is given by*

$$Gini = \frac{\theta\,(1-\theta)\left(\frac{L_r}{L_p}-1\right)}{\theta(\frac{L_r}{L_p}-1)+1}. \tag{5}$$

Proof. See the appendix. Q.E.D.

Observe from (5) that the Gini coefficient is always smaller than $1-\theta$ and is also strictly increasing in $\frac{L_r}{L_p}$. Moreover, Gini coefficient increases with θ when $\theta \in (0, \frac{\sqrt{L_p}}{\sqrt{L_r}+\sqrt{L_p}})$ and decreases with it when $\theta \in (\frac{\sqrt{L_p}}{\sqrt{L_r}+\sqrt{L_p}}, 1)$. When $\theta = \frac{\sqrt{L_p}}{\sqrt{L_r}+\sqrt{L_p}}$, the society reaches the maximum level of inequality with $Gini_{\max} = \frac{\sqrt{L_r}-\sqrt{L_p}}{\sqrt{L_r}+\sqrt{L_p}}$. Alternatively, when and only when $\frac{L_r}{L_p} < \left(\frac{1-\theta}{\theta}\right)^2$, Gini coefficient increases with θ.

Market Equilibrium

Let p_m and p_a denote the market prices for non-agricultural good and the agriculture good, respectively, then perfect competition implies

$$p_m = \frac{W}{A_m},\ p_a = \frac{W}{A_a}. \tag{6}$$

Consider a consumer who wants to maximize her utility function (1) subject to the following budget constraint:

$$p_m c_m + p_a c_a \leq I, \tag{7}$$

where income $I \in \{I_r, I_p\}$ given by (4).
This yields the following optimal consumption c_m and c_a:

$$c_a = \begin{cases} p_a^{-\epsilon} p_m^{\epsilon}, & \text{if} \quad I \geq p_a^{1-\epsilon} p_m^{\epsilon} \\ \frac{I}{p_a}, & \text{otherwise} \end{cases}, \tag{8}$$

and

$$c_m = \begin{cases} \frac{I - p_a^{1-\epsilon} p_m^{\epsilon}}{p_m}, & \text{if} \quad I \geq p_a^{1-\epsilon} p_m^{\epsilon} \\ 0, & \text{otherwise} \end{cases}. \tag{9}$$

Substituting (8) and (9) into (1) yields the real income (or final consumption) as follows:

$$c = \begin{cases} \frac{I}{p_m} + \frac{1}{\epsilon - 1} \left(p_a^{-1} p_m \right)^{\epsilon - 1}, & \text{if} \quad I \geq p_a^{1-\epsilon} p_m^{\epsilon} \\ \frac{\epsilon}{\epsilon - 1} \left(\frac{I}{p_a} \right)^{\frac{\epsilon - 1}{\epsilon}}, & \text{otherwise} \end{cases},$$

which, together with (4) and (6), implies that the real income of a household with human capital $L \in \{L_r, L_p\}$ is

$$c = \begin{cases} A_m L + \frac{1}{\epsilon - 1} \left(\frac{A_a}{A_m} \right)^{\epsilon - 1}, & \text{if} \quad L \geq A_m^{-\epsilon} A_a^{\epsilon - 1} \\ \frac{\epsilon}{\epsilon - 1} \left(A_a L_p \right)^{\frac{\epsilon - 1}{\epsilon}}, & \text{otherwise} \end{cases}. \tag{10}$$

Discussion It is analytically isomorphic to interpret (1) as the production function of the final consumption good, which is produced by combining two intermediate inputs: agriculture good and non-agriculture good. The non-homotheticity of (1) implies that income distribution matters for both aggregate demand and aggregate price levels given the non-negativity constraint on c_m. Moreover, (1) is of decreasing returns to scale when interpreted as a production function, so the more spread the production scale, the better. The natural minimum scale of production is at the household level, which is equivalent to the problem of household utility maximization when (1) is interpreted as part of the utility function. Without loss of generality, normalize the price of final consumption good defined in (1) to unity. There are two advantages to choose the final consumption good as numeraire. First, GDP and welfare will be in the same unit, which enormously simplifies the welfare analysis. Second, it is easier than other choices of numeraire to conduct GDP analyses with or without policy interventions, given the non-homotheticity of (1) with the potential binding non-negativity constraint on c_m.

Next we explore three different scenarios depending on whether $A_m^{-\epsilon} A_a^{\epsilon - 1}$ is inside or outside the interval (L_p, L_r).

Scenario I. Only Rich Households Consume Non-agriculture.

Suppose the following is true

$$L_r > A_m^{-\epsilon} A_a^{\epsilon - 1} \geq L_p. \tag{11}$$

In this case, (10) implies that the total GDP is given by

$$Y=\theta\left[A_mL_r+\frac{1}{\epsilon-1}\left(\frac{A_a}{A_m}\right)^{\epsilon-1}\right]+(1-\theta)\frac{\epsilon}{\epsilon-1}\left(A_aL_p\right)^{\frac{\epsilon-1}{\epsilon}}, \tag{12}$$

where the first term on the right hand side is the total income of rich households whereas the second term is the total income of poor households. Since only rich households can afford non-agriculture good, so the aggregate demand for non-agriculture good and agriculture is given by:

$$D_m = \theta\left(A_mL_r-A_a^{\epsilon-1}A_m^{1-\epsilon}\right); \tag{13}$$

$$D_a = \theta\left(\frac{A_a}{A_m}\right)^{\epsilon}+(1-\theta)A_aL_p. \tag{14}$$

The equilibrium amount of human capital used in the non-agriculture and agriculture sectors (denoted by L_m and L_a, respectively) are, respectively, given by

$$L_m=\frac{D_m}{A_m}=\theta\left(L_r-A_a^{\epsilon-1}A_m^{-\epsilon}\right), \tag{15}$$

and

$$L_a=\frac{D_a}{A_a}=\theta A_a^{\epsilon-1}A_m^{-\epsilon}+(1-\theta)L_p. \tag{16}$$

The value added share (equivalent to human capital share) of the non-agriculture sector (denoted by η_m) in the whole economy is given by

$$\eta_m\equiv\frac{L_m}{L_a+L_m}=\frac{\theta\left(L_r-A_a^{\epsilon-1}A_m^{-\epsilon}\right)}{(1-\theta)L_p+\theta L_r}. \tag{17}$$

Obviously,

$$\frac{\partial\eta_m}{\partial L_r}>0;\ \frac{\partial\eta_m}{\partial L_p}<0;\ \frac{\partial\eta_m}{\partial\theta}>0;\ \frac{\partial\eta_m}{\partial A_m}>0;\ \frac{\partial\eta_m}{\partial A_a}<0.$$

We assume throughout this paper that the non-agriculture sector gives priority in employing workers with high human capital. Let N_m denote the employment share in the non-agriculture sector, which is equal to the total head account of workers in that sector because the total measure of workers is unity). Then (15) implies

$$N_m=\theta\left(1-\frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}\right), \tag{18}$$

so a measure of $\theta\frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}$ workers with high human capital and all the workers with low human capital are employed in the agriculture sector. In this economy, all workers for the agriculture sector live in the rural region while all workers for the non-agriculture sector live in the urban region.

The Gini coefficient in the urban region is zero because all residents are rich households (namely, households with human capital L_r), whereas the Gini coefficient in the rural region can be computed as

$$GINI_r = \frac{\theta(1-\theta)\left(1-\frac{L_p}{L_r}\right)A_a^{\epsilon-1}A_m^{-\epsilon}}{\left[L_p+\theta\left(A_a^{\epsilon-1}A_m^{-\epsilon}-L_p\right)\right]\left[1-\theta+\theta\frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}\right]}, \tag{19}$$

the proof of which is delegated to the appendix.

Scenario II All Households Consume Non-agriculture.

When the following is true

$$L_r > L_p \geq A_m^{-\epsilon}A_a^{\epsilon-1}, \tag{20}$$

(10) implies that both rich and poor households can afford to consume non-agriculture good and the total GDP is

$$Y = A_m\left\{\theta L_r + (1-\theta)L_p + \frac{1}{\epsilon-1}A_a^{\epsilon-1}A_m^{-\epsilon}\right\}. \tag{21}$$

The aggregate demand for non-agriculture good and agriculture is given by:

$$\begin{aligned} D_m &= A_m\left[\theta L_r + (1-\theta)L_p\right] - A_a^{\epsilon-1}A_m^{1-\epsilon}, \\ D_a &= \left(\frac{A_a}{A_m}\right)^{\epsilon}, \end{aligned}$$

and the human capital allocated to the non-agriculture and agriculture sectors is respectively given by:

$$L_m = \frac{D_m}{A_m} = \left[\theta L_r + (1-\theta)L_p\right] - A_a^{\epsilon-1}A_m^{-\epsilon}, \tag{22}$$

and

$$L_a = A_a^{\epsilon-1}A_m^{-\epsilon}. \tag{23}$$

The value added share of the non-agriculture sector is

$$\eta_m = \frac{L_m}{L_a+L_m} = \frac{\left[\theta L_r + (1-\theta)L_p\right] - A_a^{\epsilon-1}A_m^{-\epsilon}}{(1-\theta)L_p+\theta L_r}. \tag{24}$$

Obviously

$$\frac{\partial \eta_m}{\partial L_r} > 0; \frac{\partial \eta_m}{\partial L_p} > 0; \frac{\partial \eta_m}{\partial \theta} > 0; \frac{\partial \eta_m}{\partial A_m} > 0; \frac{\partial \eta_m}{\partial A_a} < 0.$$

The employment share of the agriculture sector is

$$N_a = \begin{cases} \frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_p}, & \text{if } A_a^{\epsilon-1}A_m^{-\epsilon} < (1-\theta)L_p \\ \frac{(1-\theta)[L_r-L_p]+A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}, & \text{if } (1-\theta)L_p \leq A_a^{\epsilon-1}A_m^{-\epsilon} \leq L_p \end{cases},$$

and the employment share of the non-agriculture sector is given by

$$N_m = \begin{cases} 1 - \frac{A_a^{\epsilon-1} A_m^{-\epsilon}}{L_p}, & \text{if } A_a^{\epsilon-1} A_m^{-\epsilon} < (1-\theta) L_p \\ \frac{[\theta L_r + (1-\theta) L_p] - A_a^{\epsilon-1} A_m^{-\epsilon}}{L_r}, & \text{if } (1-\theta) L_p \leq A_a^{\epsilon-1} A_m^{-\epsilon} \leq L_p \end{cases}, \quad (25)$$

which increases with L_p and is independent of θ. We can derive the rural Gini coefficient as follows:

$$GINI_r = \begin{cases} \frac{\left[A_a^{\epsilon-1} A_m^{-\epsilon} - (1-\theta) L_p\right]\left(1 - \frac{L_p}{L_r}\right)(1-\theta)}{A_a^{\epsilon-1} A_m^{-\epsilon}\left[(1-\theta) + \frac{A_a^{\epsilon-1} A_m^{-\epsilon} - (1-\theta) L_p}{L_r}\right]}, & \text{if } (1-\theta) L_p \leq A_a^{\epsilon-1} A_m^{-\epsilon} \leq L_p \\ 0, & \text{if } A_a^{\epsilon-1} A_m^{-\epsilon} < (1-\theta) L_p \end{cases}, \quad (26)$$

and the urban Gini coefficient is given as

$$GINI_u = \begin{cases} 0, & \text{if } (1-\theta) L_p \leq A_a^{\epsilon-1} A_m^{-\epsilon} \leq L_p \\ \frac{\theta (L_r - L_p)\left[1 - \theta - \frac{A_a^{\epsilon-1} A_m^{-\epsilon}}{L_p}\right]}{\left[1 - \frac{A_a^{\epsilon-1} A_m^{-\epsilon}}{L_p}\right]\left[\left(1 - \theta - \frac{A_a^{\epsilon-1} A_m^{-\epsilon}}{L_p}\right) L_p + \theta L_r\right]}, & \text{if } A_a^{\epsilon-1} A_m^{-\epsilon} < (1-\theta) L_p \end{cases}.$$

Scenario III No Households Consume Non-agriculture.

When the following is true

$$A_m^{-\epsilon} A_a^{\epsilon-1} \geq L_r > L_p, \quad (27)$$

(10) implies that no household can afford to consume non-agriculture goods, and the total GDP is given by

$$Y = \theta \frac{\epsilon}{\epsilon - 1} (A_a L_r)^{\frac{\epsilon-1}{\epsilon}} + (1-\theta) \frac{\epsilon}{\epsilon - 1} (A_a L_p)^{\frac{\epsilon-1}{\epsilon}}, \quad (28)$$

so the aggregate demand for non-agriculture good and agriculture is

$$D_m = 0; \; D_a = A_a \left[\theta L_r + (1-\theta) L_p\right].$$

All labor is employed in the agriculture sector and no industrialization occurs:

$$\eta_m = N_m = 0 \quad (29)$$

The rural Gini coefficient is the same as the Gini coefficient for the whole economy, given by (5).

Summary

Based on the analyses above for the three different scenarios, we summarize the total GDP, sectorial value-added shares and employment shares, and rural Gini coefficient in the market equilibrium. Define

$$\Omega \equiv A_m^{-\epsilon} A_a^{\epsilon-1}. \quad (30)$$

Proposition 2. *The endogenous aggregate production function, denoted by $F(L_r, L_p, A_m, \Omega)$, has the following functional forms:*

$$F(L_r, L_p, A_m, \Omega) = \begin{cases} \frac{\epsilon}{\epsilon-1} A_m \Omega^{\frac{1}{\epsilon}} \left[\theta L_r^{\frac{\epsilon-1}{\epsilon}} + (1-\theta) L_p^{\frac{\epsilon-1}{\epsilon}}\right] & \Omega > L_r \\ A_m \left[\theta \left(L_r + \frac{1}{\epsilon-1}\Omega\right) + (1-\theta) \frac{\epsilon}{\epsilon-1} \Omega^{\frac{1}{\epsilon}} L_p^{\frac{\epsilon-1}{\epsilon}}\right], & \text{if } L_p \leq \Omega \leq L_r \\ A_m \left[\theta L_r + (1-\theta) L_p + \frac{1}{\epsilon-1}\Omega\right], & \text{if } \Omega < L_p \end{cases}, \quad (31)$$

and the equilibrium real wage rate per unit of human capital is

$$W = \frac{F(L_r, L_p, A_m, \Omega)}{\theta L_r + (1-\theta) L_p},$$

where Ω is defined in (30)
Proof. Combine (12), (21) and (28) and use (30). Q.E.D.

This proposition characterizes how total GDP Y and real wage rate per unit of human capital change with L_r, L_p, A_m and Ω. When holding A_m fixed, total GDP Y as a function of Ω is illustrated in the following figure.

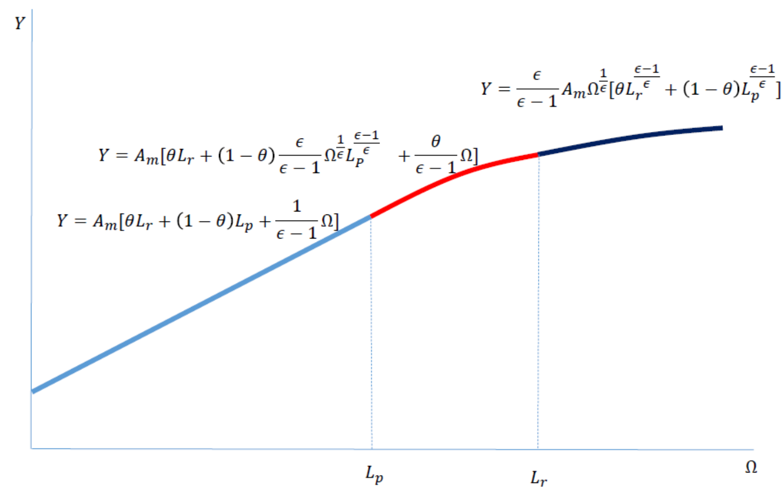

Figure 1. GDP as a function of Ω

Clearly, the aggregate production function (expression for GDP) strictly increases with Ω but has different functional forms when Ω is on different intervals, which reflects the fact that the underlying economic structures are endogenously different for the three different scenarios analyzed above. It is in fact a common technical feature of models in New Structural Economics (see Ju, Lin and Wang (2015), Lin and Wang (2018)). Interestingly, observe from (31) that when $\Omega > L_r$, the aggregate production function is a CES aggregate of human capital endowment of a rich household L_r and that of a poor household L_p, with substitution elasticity equal to ϵ, the price demand elasticity for agriculture products. When $L_p \leq \Omega \leq L_r$, the aggregate production function is a quasi-linear function of L_r and L_p up to an additive term that only depends on productivities. When $\Omega < L_p$ the aggregate production function is a linear function of L_r and L_p up to an additive term. Note that $A_a = A_m \Omega^{\frac{1}{\epsilon}}$, so when $\Omega > L_r$, the total GDP is independent of A_m.

Due to the non-homothetic preference, households with different income levels have different consumption structures (that is, $\frac{c_a}{c_m}$ is different), so the production structures are also different, depending on the income heterogeneity across households. When Ω changes, both household incomes and prices change, so the household heterogeneity and non-homothetic preference jointly determine the aggregate demand for agriculture and non-agriculture products, resulting in structural changes and changes in the functional form of the aggregate production function. The equilibrium wage rate in terms of the final good W also depends on the income heterogeneity.

To see the structural change more clearly, we have the following proposition.

Proposition 3. *The value-added share of the non-agriculture sector η_m is as follows:*

$$\eta_m = \begin{cases} 0, & \Omega > L_r \\ \frac{\theta(L_r-\Omega)}{(1-\theta)L_p+\theta L_r}, & \text{if } L_p \leq \Omega \leq L_r \\ \frac{\theta L_r+(1-\theta)L_p-\Omega}{(1-\theta)L_p+\theta L_r}, & \text{if } \Omega < L_p \end{cases}, \tag{32}$$

and the employment share of the non-agriculture sector N_m is given by

$$N_m = \begin{cases} 0, & \Omega > L_r \\ \theta\left(1-\frac{\Omega}{L_r}\right), & \text{if } L_p \leq \Omega \leq L_r \\ \frac{\theta L_r+(1-\theta)L_p-\Omega}{L_r}, & \text{if } (1-\theta)L_p \leq \Omega < L_p \\ 1-\frac{\Omega}{L_p} & \text{if } \Omega \leq (1-\theta)L_p \end{cases}, \tag{33}$$

where Ω is defined in (30).

Proof. Combing (17),(24) and (29) yields (32). Combining (18), (25) and (29) yields (33). Q.E.D.

More intuitively, Figure 2 shows how the value-added share of the non-agriculture sector η_m changes with productivities Ω and Figure 3 plots how employment share N_m changes with Ω.

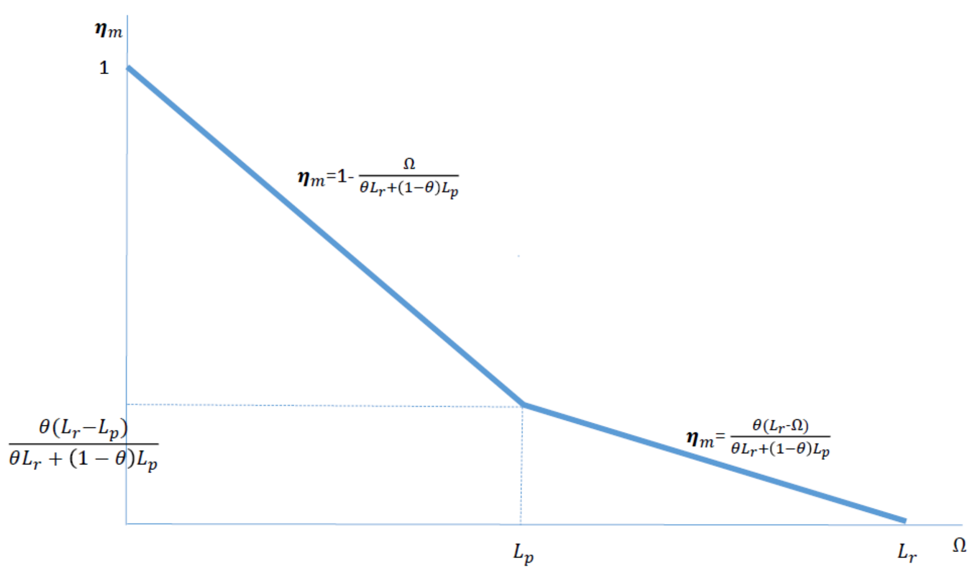

Figure 2. How Value-added Share of Non-Agriculture Sector Changes with Ω.

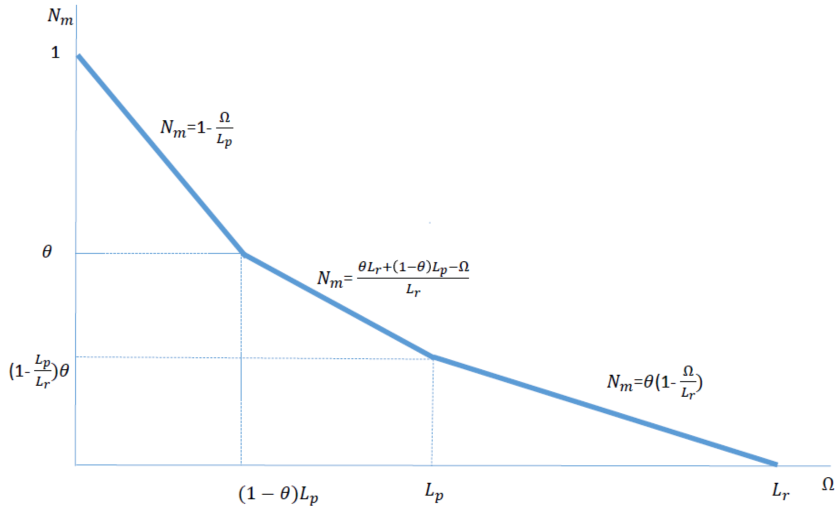

Figure 3. How Employment Share of Non-Agriculture Sector Changes with Ω.

Observe that value-added share η_m and employment share N_m are different because workers are heterogeneous in human capital endowment. More explicitly, when $\Omega \leq (1-\theta)L_p$, all agriculture output will be produced by $\frac{\Omega}{L_p}$ poor households (or workers with low human capital), the remaining $1-\theta-\frac{\Omega}{L_p}$ poor households and all θ rich households will work in the non-agriculture sector. When $(1-\theta)L_p < \Omega \leq L_p$, the agriculture output will be produced by all $1-\theta$ poor households plus $\frac{\Omega-(1-\theta)L_p}{L_r}$ rich households, and the remaining $\theta - \frac{\Omega-(1-\theta)L_p}{L_r}$ rich households will work in the non-agriculture sector. When $L_p < \Omega \leq L_r$, the agriculture output will be produced by all $1-\theta$ poor households and $\frac{\theta\Omega}{L_r}$ rich households, and the remaining $\theta - \frac{\theta\Omega}{L_p}$ rich households will work in the non-agriculture sector. When $\Omega > L_r$, no one can afford to consume non-agriculture good, and all households will be working in the agriculture sector. These different regimes could explain why there are kinks in Figure 3. The reason why there is a kink when $\Omega = L_p$ in Figure 2 is because the aggregate (induced) demand for agricultural labor jumps down from Ω to $\theta\Omega$ once Ω crosses the threshold value L_p from below as the non-agriculture products suddenly become too expensive for poor households to consume, so only rich households, which account for θ fraction of the population, will each consume non-agriculture products with the amount produced by Ω units of human capital.

Using the definition of Gini coefficient, we can derive the rural Gini coefficient, which is summarized in the following proposition.

Proposition 4 *The Gini coefficient in the rural region is given by*

$$GINI_r = \begin{cases} \frac{\theta(1-\theta)\left(\frac{L_r}{L_p}-1\right)}{\theta(\frac{L_r}{L_p}-1)+1} & \Omega \geq L_r \\ \frac{\theta(1-\theta)\left(1-\frac{L_p}{L_r}\right)\Omega}{[L_p+\theta(\Omega-L_p)]\left[1-\theta+\theta\frac{\Omega}{L_r}\right]} & \text{if } L_p \leq \Omega < L_r \\ \frac{[\Omega-(1-\theta)L_p]\left(1-\frac{L_p}{L_r}\right)(1-\theta)}{\Omega\left[(1-\theta)+\frac{\Omega-(1-\theta)L_p}{L_r}\right]}, & \text{if } (1-\theta)L_p \leq \Omega < L_p \\ 0, & \text{if } \Omega < (1-\theta)L_p \end{cases}, \tag{34}$$

where Ω is defined in (30).

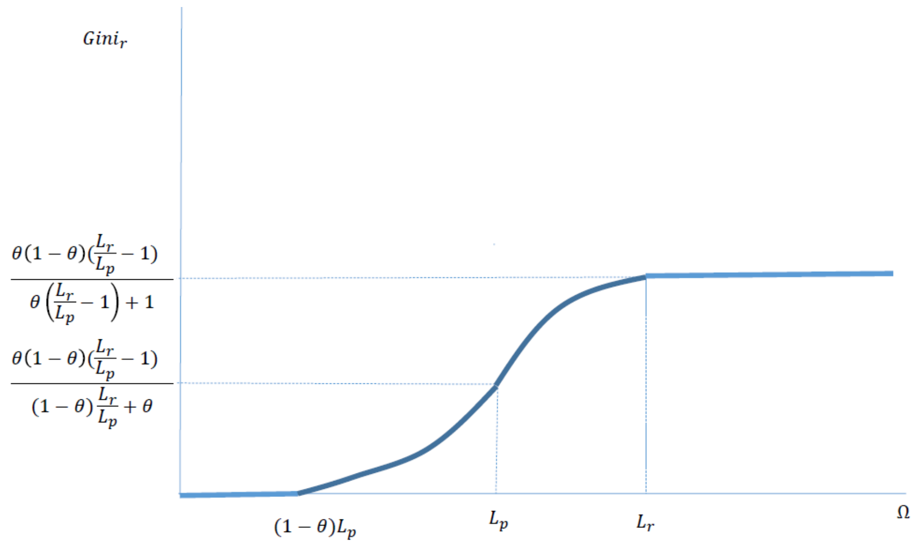

Figure 1: Figure 4a. Rural Gini Coefficient when $L_r \leq \left(\frac{1-\theta}{\theta}\right)^2 L_p$ and $\theta < \frac{1}{2}$.

Proof. Combine (19), (26) and (5), and use (30). Q.E.D.
It turns out that $\frac{\partial GINI_r}{\partial \Omega} > 0$ when $(1-\theta)L_p \leq \Omega < L_p$. However, $GINI_r$ could change with Ω non-monotonically when $L_p \leq \Omega < L_r$. More specifically, when $L_r \leq \left(\frac{1-\theta}{\theta}\right)^2 L_p$ and $\theta < \frac{1}{2}$, $GINI_r$ strictly increases with Ω for any $(1-\theta)L_p \leq \Omega < L_r$. This is plotted in Figure 4a. The intuition is as follows. All workers are employed in the rural sector when $\Omega \geq L_r$, so the rural Gini coefficient is the same as the Gini coefficient for the whole economy, which is given by (5). Now suppose Ω decreases so that $L_p < \Omega < L_r$ holds, we learn from the previous proposition that workers with high human capital move from the agriculture sector into the non-agriculture sector. Note that the rich households are minority in the rural region ($\theta < \frac{1}{2}$) and $\frac{L_r}{L_p}$ is small enough ($L_r \leq \left(\frac{1-\theta}{\theta}\right)^2 L_p$), so when rich households leave the rural region, it is as if θ decreases, so the Gini coefficient decreases and the income distribution in the rural region is becoming more equalized.
However, when $L_r > \left(\frac{1-\theta}{\theta}\right)^2 L_p$ and $\theta < \frac{1}{2}$, the rural Gini coefficient is plotted in Figure 4b, where $\widetilde{L} \equiv \frac{1-\theta}{\theta}\sqrt{L_p L_r}$. Observe that $GINI_r$ increases with Ω when $\Omega \in (L_p, \widetilde{L})$ and decreases with Ω when $\Omega \in (\widetilde{L}, L_r]$. The rural Gini coefficient $GINI_r$ reaches the maximum value $\frac{\sqrt{L_r}-\sqrt{L_p}}{\sqrt{L_r}+\sqrt{L_p}}$ when $\Omega = \widetilde{L}$.

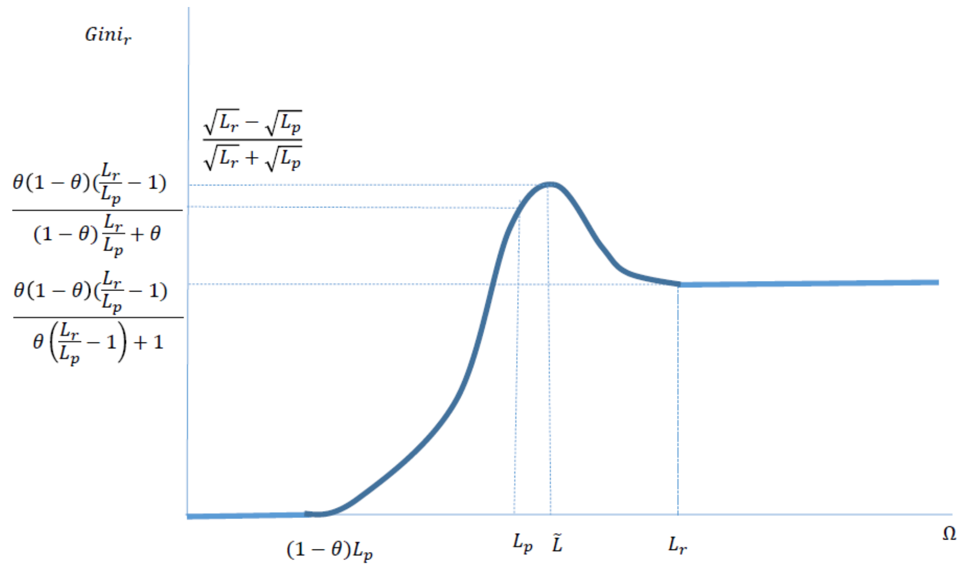

Figure 2: Figure 4c. Rural Gini Coefficient when $L_r > \left(\frac{\theta}{1-\theta}\right)^2 L_p$ and $\theta \geq \frac{1}{2}$.

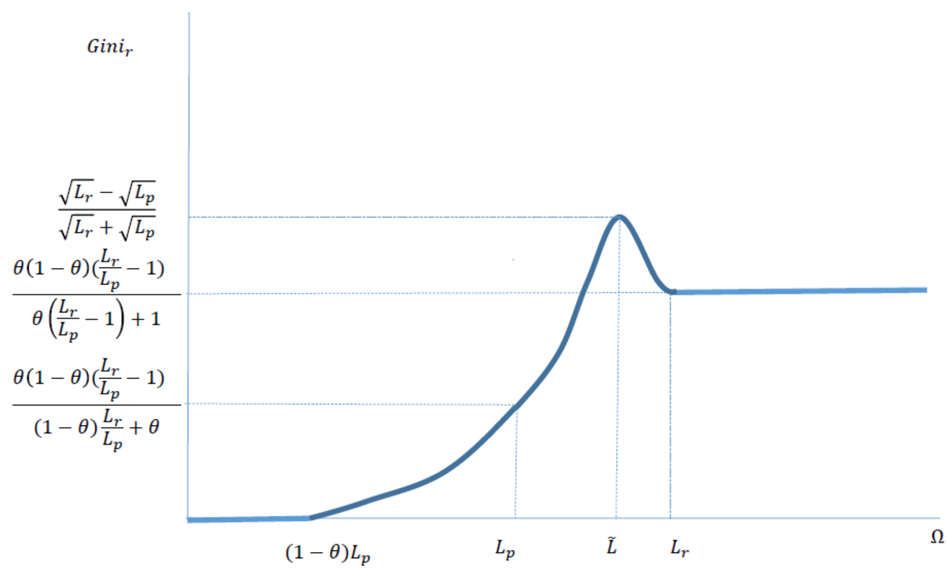

Figure 4b. Rural Gini Coefficient when $L_r > \left(\frac{1-\theta}{\theta}\right)^2 L_p$ and $\theta < \frac{1}{2}$.

Similarly, when $\theta \geq 1/2$ and $L_r > \left(\frac{\theta}{1-\theta}\right)^2 L_p$, $GINI_r$ increases with Ω when $\Omega \in (L_p, \widetilde{L})$ and decreases with Ω when $\Omega \in (\widetilde{L}, L_r]$. This case is plotted in Figure 4c. When $\theta \geq 1/2$ and $L_r \leq \left(\frac{\theta}{1-\theta}\right)^2 L_p$, $GINI_r$ decreases with Ω for any $\Omega \in (L_p, L_r]$. This is shown in Figure 4d. We leave the proof of how $GINI_r$ changes with Ω in the appendix.

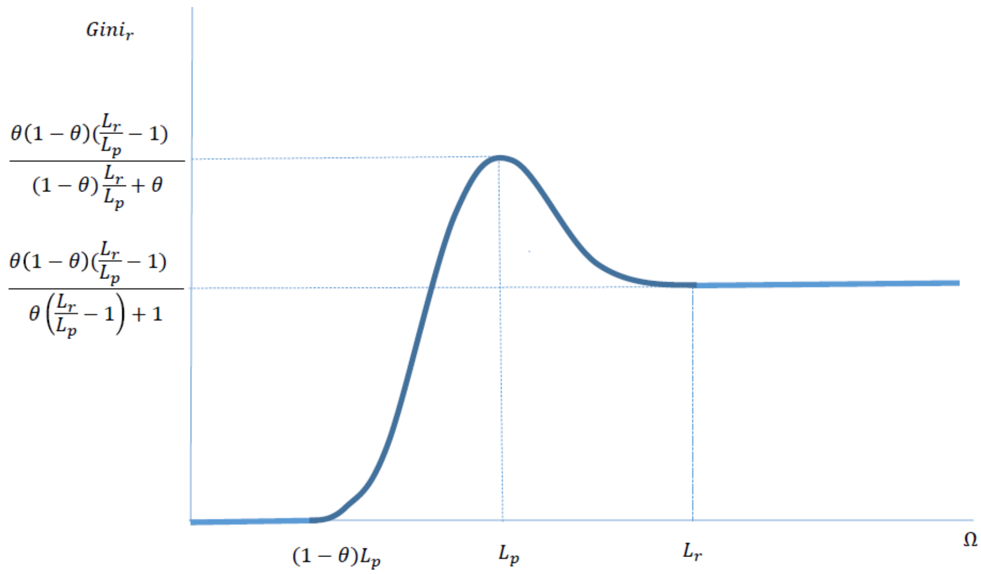

Figure 4d. Rural Gini Coefficient when $L_r \leq \left(\frac{\theta}{1-\theta}\right)^2 L_p$ and $\theta \geq \frac{1}{2}$.

In this part, productivities A_m and A_a are exogenous. Next, we make the model dynamic by allowing A_m and A_a to change endogenously over time.

Dynamic Model

Consider a continuous-time infinite-horizon world, where households' utility function is given by

$$\int_0^\infty u(c(t))e^{-\rho t}dt, \tag{35}$$

where ρ is the time discount rate and is strictly positive and (1) holds for each time point t. For simplicity, assume all goods are non-storable. Suppose productivities in the two sectors evolve as follows:

$$\dot{A}_m = A_m L_m^\alpha; \;\; \dot{A}_a = A_a L_a^\alpha, \tag{36}$$

where $0 < \alpha < 1$. That is, as more effective units of labor is employed to produce in a sector, the productivity of that sector increases due to learning by doing.
Using the definition of Ω in (30) and (**??**), we obtain

$$\frac{\dot{\Omega}}{\Omega} \leq 0 \Leftrightarrow \left[\frac{\epsilon - 1}{\epsilon}\right]^{\frac{1}{\alpha}} \leq \frac{L_m}{L_a}, \tag{37}$$

where $\frac{\dot{\Omega}}{\Omega} = 0$ if and only if $\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} = \frac{L_m}{L_a}$ holds. Consider the decentralized competitive market equilibrium, in which each household maximizes (35) subject to (7) by choosing which sector to work, how much labor to supply, and $c_m(t)$ and $c_a(t)$ for all time $t \in [0, \infty)$ by taking all prices as exogenously given. When (11) is satisfied, substituting (15) and (16) into (37) yields $\frac{\dot{\Omega}}{\Omega} < 0$ if and only if $L_p \leq \Omega < L^*$, where

$$L^* \equiv \frac{L_r - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} (\frac{1}{\theta} - 1) L_p}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1}. \tag{38}$$

Obviously, $L^* < L_r$. Moreover, $L^* \geq L_p$ if and only if

$$L_r \geq \left[1 + \left[\frac{\epsilon - 1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right] L_p. \tag{39}$$

When (20) is true, substituting (22) and (23) to (37) yields $\frac{\dot{\Omega}}{\Omega} < 0 \Leftrightarrow \Omega < L^{**}$, where

$$L^{**} \equiv \frac{\theta L_r + (1-\theta) L_p}{1 + \left[\frac{(\epsilon-1)}{\epsilon}\right]^{\frac{1}{\alpha}}}. \tag{40}$$

It turns out that $L^{**} \leq L^*$ if and only if (39) holds. Moreover, $L^{**} \geq L_p$ if and only if (39) holds. When (27) is true, $L_m = 0$ and so (37) implies $\frac{\dot{\Omega}}{\Omega} > 0$. These findings are summarized in the following lemma.

Lemma 1. *When (39) is true,* $\dot{\Omega} < 0$ *if and only if* $\Omega \in [0, L^*)$*, and* $\dot{\Omega} > 0$ *if and only if* $\Omega \in (L^*, \infty)$*. When (39) is violated,* $\dot{\Omega} < 0$ *if and only if* $\Omega \in [0, L^{**})$*, and* $\dot{\Omega} > 0$ *if and only if* $\Omega \in (L^{**}, \infty)$*.*

Proposition 5. *Suppose (39) is true. There exists a balanced growth path (referred as steady state 1 henceforth), on which all rich households consume both agriculture and non-agriculture goods and all poor households only consume agriculture goods. Moreover, the value-added share of the non-agriculture sector is*

$$\eta_m = \frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1}, \tag{41}$$

the employment share is $N_m = \frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1} \frac{\theta L_r + (1-\theta) L_p}{L_r}$*, and the growth rate of total GDP, denoted by* g_{GDP}*, is given by*

$$g_{GDP} = \frac{\dot{A}_m}{A_m} = \frac{\epsilon - 1}{\epsilon} \left(\frac{\theta L_r + (1-\theta) L_p}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1}\right)^{\alpha}, \tag{42}$$

and the Gini coefficient in the urban region is zero whereas the Gini coefficient in the rural region is

$$GINI_r = \frac{(1-\theta)\left(\frac{L_r}{L_p} - 1\right)\left[\theta \frac{L_r}{L_p} - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} (1-\theta)\right]\left[\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1\right]}{\left[(1-\theta) + \theta \frac{L_r}{L_p}\right]\left[\frac{L_r}{L_p} + (1-\theta)\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\left(\frac{L_r}{L_p} - 1\right)\right]},$$

and $\Omega = L^*$, where L^* is given by (38).
Proof. See the Appendix. Q.E.D.
On this Balanced Growth Path (BGP), the value added of the two sectors grows at the same speed as total GDP, and there is no structural change (labor reallocation across the two sectors). Observe that when price demand elasticity ϵ increases, value-added share and employment share of the non-agriculture sector both increase ($\frac{\partial \eta_m}{\partial \epsilon} > 0$, $\frac{\partial N_m}{\partial \epsilon} > 0$), so does the GDP growth rate ($\frac{\partial g_{GDP}}{\partial \epsilon} > 0$). Moreover, when α increases, value-added share and employment share of the non-agriculture sector both become larger ($\frac{\partial \eta_m}{\partial \alpha} > 0$, $\frac{\partial N_m}{\partial \alpha} > 0$). Notice that η_m is independent of θ, L_p and L_r on the BGP, but the opposite is true in Scenario I in the static model. Both N_m and g_{GDP} increase with θ and L_p, whereas an increase in L_r reduces N_m but increases g_{GDP}.

Lemma 1 implies that the BGP (on which $\Omega = L^*$ holds) characterized in the last proposition is unstable. The following proposition characterizes what happens off the BGP.
Proposition 6. *Suppose (39) is true. When $\Omega(0) < L^*$ holds, the economy will keep industrializing, GDP will grow monotonically, and the economy will converge to an asymptotic steady state (referred as asymptotic steady state 2 henceforth), in which all households consume both agriculture (negligible) and non-agriculture goods and the following is true:*

$$\begin{aligned} \dot{A}_m &= A_m \left[\theta L_r + (1-\theta)L_p\right]^{\alpha}, \\ \dot{A}_a &= 0, \\ g_{GDP} &= \left[\theta L_r + (1-\theta)L_p\right]^{\alpha}, \\ \eta_m &= N_m = 1. \end{aligned} \tag{43}$$

When $\Omega(0) > L^*$ holds, the economy will keep de-industrializing, GDP will grow monotonically, and the economy will converge to the agrarian steady state (referred as agrarian steady state 3 henceforth), in which all households consume agriculture goods only and the following is true :

$$\begin{aligned} \dot{A}_m &= 0, \\ \dot{A}_a &= A_a \left[\theta L_r + (1-\theta)L_p\right]^{\alpha}, \\ g_{GDP} &= \frac{\epsilon-1}{\epsilon}\left[\theta L_r + (1-\theta)L_p\right]^{\alpha} \\ \eta_m &= N_m = 0. \end{aligned} \tag{44}$$

Proof. Using (31), (**??**), (32), (33) and Lemma 1. Note that both A_a and A_m will always weakly increase and at least one of them strictly increases any time, so GDP will keep increasing strictly. Q.E.D.

Observe that the GDP growth rate is highest in steady state 2, second highest in steady state 3, and lowest in steady state 1. Suppose $\Omega \in [L_p, L^*)$ initially. The economy starts with Scenario I characterized in the static model, in which rich households consume both agriculture and non-agriculture goods and poor households only consume agriculture goods.

Then Ω monotonically decreases over time, labor continuously moves from the agriculture sector into the non-agriculture sector and the value added share of the non-agriculture sector keeps increasing. When $\Omega < L_p$, poor households also consume both agriculture and non-agriculture goods. This industrialization process lasts forever, converging to the asymptotic steady state (steady state 3), in which all people work in the non-agriculture sector and no agriculture goods will be consumed. Suppose, on the other hand, $\Omega \in (L^*, L_r)$ holds initially.

Then de-industralization will take place continuously till the economy reaches steady state 3, in which every household only consumes agriculture goods and nobody works in the non-agriculture sector.
How does the rural Gini coefficient change over time ? Suppose (39) is true and $\theta < \frac{1}{2}$. When $\theta \in (0, \frac{1}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+2})$, Figure 4a applies. That is, when $\Omega(0) < L^*$, the rural Gini coefficient monotonically decreases as Ω declines over time till Ω reaches $(1-\theta)L_p$, after which the rural Gini coefficient is always zero. When $\Omega(0) > L^*$, the rural Gini coefficient monotonically increases over time till Ω reaches L_r, after which the rural Gini coefficient remains constant at the level given by (5). When $\theta \in (\frac{1}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+2}, \frac{1}{2})$, Figure 4b applies. Moreover, $\widetilde{L} \equiv \frac{1-\theta}{\theta}\sqrt{L_p L_r} > L^*$ if and only if $1+\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\frac{1}{\theta} \leq \frac{L_r}{L_p} < H$, where

$$H \equiv \frac{\left[\left(\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+1\right)\frac{1-\theta}{\theta}+\sqrt{\left[\left(\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+1\right)\frac{1-\theta}{\theta}\right]^2+4\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\left(\frac{1}{\theta}-1\right)}\right]^2}{4}. \tag{45}$$

In that case, $\Omega(0) \in (L^*, \widetilde{L})$, the rural Gini coefficient first strictly increases over time till it reaches the maximum level $\frac{\sqrt{L_r}-\sqrt{L_p}}{\sqrt{L_r}+\sqrt{L_p}}$ when $\Omega = \widetilde{L}$, after which the rural Gini coefficient declines over time till Ω reaches L_r, after which the rural Gini remains constant at level given by (5). Similar analyses can be made when $\frac{L_r}{L_p} \geq H$ or when $\theta \geq \frac{1}{2}$. Please refer to the appendix for more details of the proof.

We can easily obtain the following two propositions when (39) is not satisfied.

Proposition 7. *Suppose (39) is violated, that is, $L_p < L_r < \left[1+\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\frac{1}{\theta}\right]L_p$. There exists a steady state, in which all households consume both agriculture and non-agriculture goods and*

$$\begin{aligned} \Omega &= L^{**}, \\ L_m &= \theta L_r + (1-\theta)L_p - L^{**}, \\ L_a &= L^{**}, \end{aligned}$$

the value-added share of the non-agriculture sector η_m is still given by (41), and the employment share is given by

$$N_m = \begin{cases} \frac{\left[\frac{(\epsilon-1)}{\epsilon}\right]^{\frac{1}{\alpha}}-\theta\left(\frac{L_r}{L_p}-1\right)}{1+\left[\frac{(\epsilon-1)}{\epsilon}\right]^{\frac{1}{\alpha}}}, & \text{if } L^{**} \leq (1-\theta)L_p \\ \frac{\left[\frac{(\epsilon-1)}{\epsilon}\right]^{\frac{1}{\alpha}}}{1+\left[\frac{(\epsilon-1)}{\epsilon}\right]^{\frac{1}{\alpha}}}\frac{\theta L_r+(1-\theta)L_p}{L_r}, & \text{if } L^{**} \in ((1-\theta)L_p, L_p) \end{cases},$$

and the growth rate of total GDP is given by (42), where L^{**} is given by (40).

Proof. Similar to that of Proposition 5. Q.E.D.

Observe further that $L^{**} \leq (1-\theta)L_p$ if and only if the human capital endowment is sufficiently close between a rich household and a poor household, or, more precisely, $L_p < L_r \leq \left[\frac{(\epsilon-1)}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1-\theta}{\theta} L_p$, which is possible only when $\theta < \eta_m$ given by (41).
This steady state is also unstable and any deviation from it would result in continuous industrialization till almost no agriculture is produced or continuous de-industralization till only agriculture is produced, depending on whether $\Omega < L^{**}$ or $\Omega > L^{**}$. It is formally stated in the following proposition.

Proposition 8. *Suppose (39) is violated, that is, $L_p < L_r < \left[1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right] L_p$. When $\Omega < L^{**}$ holds initially, the economy will converge to asymptotic steady state 2 characterized in Proposition 6. When $\Omega > L^{**}$ holds initially, the economy will converge to agrarian steady state 3 characterized in Proposition 6.*

Proof. Similar to that of Proposition 6. Q.E.D.
How the rural Gini coefficient changes over time can be analyzed analogously and is skipped here.
Is the Laissez-faire market equilibrium Pareto efficient or socially optimal? We turn to this question now.

Redistributive Policy

First consider the static case. It turns out that when not every household can afford to consume the non-agriculture consumption, the aggregate GDP can be improved by appropriate redistributive policies. More precisely, we have the following proposition.

Proposition 9. *In the static economy, when $\theta L_r + L_p(1-\theta) \geq \Omega > L_p$, the aggregate GDP can be raised to the level*

$$Y' = A_m \left[\theta L_r + (1-\theta) L_p + \frac{1}{\epsilon - 1}\Omega\right]$$

by using the following redistributive policy: Each rich household has to pay a lump-sum tax equal to $\frac{(1-\theta)[\Omega - L_p]W}{\theta}$, and all the tax revenues are equally transferred to all poor households in a lump-sum fashion. When $\theta L_r + L_p(1-\theta) < \Omega$, the total GDP can be raised to the level

$$\frac{\epsilon}{\epsilon - 1} A_m \Omega^{\frac{1}{\epsilon}} \left[\theta L_r + (1-\theta) L_p\right]^{\frac{\epsilon-1}{\epsilon}}$$

by imposing a lump-sum tax $(1-\theta)(L_r - L_p)W$ on each rich household and equally redistributing to all poor households in a lump-sum way.
Proof. See the Appendix. Q.E.D.

We emphasize that these redistributive policies are valid only when they are expected by the public before production, otherwise it has no impact on GDP. The intuition behind the

GDP-enhancing redistributive policies is that when the constraint $c_m \geq 0$ becomes binding for some household, it is equivalent to a binding borrowing constraint that prevents firms with relatively high marginal productivity form producing at a higher level. Note that the marginal productivity of agricultural labor is diminishing and also higher than that of non-agriculture labor when $c_m \geq 0$ is binding, but the poor households cannot afford enough agriculture goods. By redistributing income from rich to poor households, the aggregate demand for agriculture goods increases, so more labor is allocated into the agricultural sector to meet the demand, which improves resource allocation and hence increases aggregate GDP. Obviously, when $\Omega \leq L_p$, the demand for agriculture goods is fully satiated, and $c_m \geq 0$ is no longer binding because all the remaining income is spent on the non-agriculture goods, the technology of which is of constant returns to scale, so redistributive policies would not improve GDP.

In short, the redistributive policies are can enhance GDP because it effectively enhances the aggregate demand. An equivalent policy intervention is that the government collects the all the tax revenues in the same way as specified in the previous proposition and spend all the revenues to purchase agriculture goods as public expenditure. This Keynesian expansionary fiscal policy with a balanced government budget turns out to have a multiplier larger than unity whenever $\Omega > L_p$.

When $\theta L_r + L_p(1-\theta) \geq \Omega > L_p$, the before-redistribution rural Gini coefficient is given by

$$GINI_r = \frac{\theta(1-\theta)\left(1-\frac{L_p}{L_r}\right)\Omega}{\left[L_p+\theta(\Omega-L_p)\right]\left[1-\theta+\theta\frac{\Omega}{L_r}\right]}$$

according to Proposition 4 and the employment share of the agriculture sector is $1-\theta+\theta\frac{\Omega}{L_r}$ according to Proposition 3. The post-redistribution rural Gini coefficient is given by

$$GINI_r' = \frac{\frac{\Omega-L_p}{L_r}\left[\frac{\theta L_r+(1-\theta)L_p-\Omega}{\theta}\right]}{\left[1+\frac{\Omega-L_p}{L_r}\right]\left[\frac{\epsilon\Omega}{\epsilon-1}\frac{L_r+\Omega-L_p}{L_r}+\frac{\Omega-L_p}{L_r}\frac{\theta L_r+(1-\theta)L_p-\Omega}{\theta}\right]},$$

and the the employment share of the agriculture sector is $(1-\theta)\left(1+\frac{\Omega-L_p}{L_r}\right)$, which is larger than that before the redistribution. Among all the rural workers, there are $1-\theta$ workers with low human capital and $\frac{(1-\theta)[\Omega-L_p]}{L_r}$ workers with high human capital. When $\theta L_r + L_p(1-\theta) < \Omega$, the post-redistribution rural Gini coefficient is zero.

In the dynamic case, recall that the GDP growth rate in the asymptotic steady state 2 is higher than any other steady state. When $L_r > \Omega > L_p$, the redistributive policies prescribed in Proposition 9 encourage workers to move into the agriculture sector to boost the instantaneous aggregate GDP, however, (36) implies that such redistributive policies would result in GDP loss in the future because these policies dynamically increase Ω and push the economy away from the asymptotic steady state 2. Consequently, there is a trade-off between current GDP and future GDP when policy makers decide labor allocation across sectors.

Whereas it is difficult to characterize the precise dynamic optimal policies for the general case due to the non-linear transitional dynamics, it is nevertheless useful to examine a few special cases. To sharpen the result, suppose $u(c)$ in (35) takes the functional form of CRRA as follows:

$$u(c)=\frac{c^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}},$$

where σ is the inter-temporal elasticity of substitution. Consider the following two policies.

Policy A: An infinitely high tax rate is imposed on the consumption of agriculture permanently.

Policy B: A prohibitive tax is permanently imposed on production of non-agriculture goods and a lump-sum tax $(1-\theta)(L_r-L_p)W$ is imposed on each rich household and then equally redistributed to all poor households in a lump-sum way every time point, so that all households have equal consumption of agriculture after redistribution.

Observe that under Policy A the economy is always in steady state 2 as described in Proposition 6 and the GDP is

$$Y_A(t)=A_m(0)\cdot[\theta L_r+(1-\theta)L_p]\,e^{[\theta L_r+(1-\theta)L_p]^{\alpha}\cdot t},\forall t\in[0,\infty),$$

By contrast, under Policy B the economy is always in the steady state 3 as described in Proposition 6 and the GDP is

$$Y_B(t)=\frac{\epsilon}{\epsilon-1}A_a(0)^{\frac{\epsilon-1}{\epsilon}}\left[\theta L_r+(1-\theta)L_p\right]^{\frac{\epsilon-1}{\epsilon}}e^{\frac{\epsilon-1}{\epsilon}[\theta L_r+(1-\theta)L_p]^{\alpha}\cdot t},\ \forall t\in[0,\infty).$$

So Policy A always yields a higher GDP growth rate than Policy B. Moreover, define

$$\Omega_1 \equiv \left[\frac{\rho-\frac{\epsilon-1}{\epsilon}\left[\theta L_r+(1-\theta)L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}{\rho-\left[\theta L_r+(1-\theta)L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}\right]^{\frac{\sigma\epsilon}{\sigma-1}}\left(\frac{L_r}{\frac{\epsilon}{\epsilon-1}\left[\theta L_r+(1-\theta)L_p\right]^{\frac{\epsilon-1}{\epsilon}}}\right)^{\epsilon},$$

$$\Omega_2 \equiv \left[\frac{\rho-\frac{\epsilon-1}{\epsilon}\left[\theta L_r+(1-\theta)L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}{\rho-\left[\theta L_r+(1-\theta)L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}\right]^{\frac{\sigma\epsilon}{\sigma-1}}\left(\frac{L_p}{\frac{\epsilon}{\epsilon-1}\left[\theta L_r+(1-\theta)L_p\right]^{\frac{\epsilon-1}{\epsilon}}}\right)^{\epsilon},$$

where time discount rate ρ is assumed sufficiently large to exclude explosive growth:

$$\rho-\left[\theta L_r+(1-\theta)L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)>0.$$

It can be shown that the following is true (Please refer to the appendix for proofs): When

$\Omega(0)<\Omega_2$, every household is strictly better off under Policy A than under Policy B. When $\Omega(0)\in(\Omega_2,\Omega_1)$, every rich household is strictly better off under Policy A than under Policy B but the opposite is true for each poor household. When $\Omega(0)>\Omega_1$, every household is strictly worse off under Policy A than under Policy B. When $\Omega(0)=\Omega_2$, every poor household feels indifferent between the two policies but every rich household strictly prefers Policy A. When $\Omega(0)=\Omega_1$, every rich household feels indifferent between the two policies but every poor household strictly prefers Policy B.

This example suggests that at which steady state the welfare of a household is higher depends on the level of initial productivities $\Omega(0)$, not necessarily the steady state that yields the higher GDP growth rate. In general, a high enough A_m or a low enough A_a would in general make Policy A more favorable than Policy B. Moreover, households with different human capital endowment may have different preferences over policies.

Suppose (39) holds and $\Omega(0)$ is equal to L^*, given by (38), so from time 0 the economy is always at the steady state characterized in Proposition 5. Is a rich household strictly better off under Policy A than in this Laissez-faire market equilibrium?

It turns out that the answer is positive if and only if the inter-temporal elasticity satisfies $\sigma \in (\sigma^*, 1)$, where σ^* is uniquely determined by

$$\log \frac{1 + \left(1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right)(\epsilon - 1)}{\left(1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right)(\epsilon - 1)} = \frac{\sigma^2}{1-\sigma} \log \left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}} \right],$$

and

$$\frac{1}{1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}} \geq \frac{L_p}{L_r} > \frac{1 - (\epsilon-1)\left(\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1\right) \left[\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1} \right]^{\frac{\sigma^2}{\sigma-1}} - 1 \right]}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \left(\frac{1}{\theta} - 1\right)}, \tag{46}$$

Please refer to the appendix to see the proof. This example suggests that inter-temporal elasticity of substitution σ, initial productivities $\Omega(0)$, and human capital heterogeneity $\frac{L_p}{L_r}$ could be all important in determining whether a household is better off in a Laissez affair market equilibrium or in a steady state under policy interventions. In this specific example, if $\sigma \leq \sigma^*$ or $\sigma \geq 1$, the Laissez affair market equilibrium characterized in Proposition 5 delivers a higher welfare level to the rich households than Policy A, even though the latter achieves full industrialization and attains a higher growth rate of GDP. Furthermore, based on the previous discussion, when $L^* < \Omega_2$, rich households are better off under Policy A than Policy B, so the Laissez affair equilibrium when $\Omega(0) = L^*$ is better than Policy B in terms of the welfare of rich households. This example shows that when human capital endowment becomes too heterogeneous in the sense that $\frac{L_p}{L_r}$ is sufficiently small so that the second inequality is violated in (46), Policy A also makes rich households worse off than in the Laissez affair market equilibrium. Similar analyses can be made on the welfare of poor households and the steady state characterized in Proposition 7.

Conclusion

In this paper, we develop a simple dynamic model of (de)industrialization and income distribution to analytically characterize how these two dynamic processes interact with each other and what they imply for aggregate GDP growth, the evolution of rural income distribution as well as welfare. Redistributive policies are shown to be sometimes useful to improve GDP via structural change. The high tractability of the model mainly comes from the new non-homothetic utility function introduced in this paper. Several avenues for future research seem appealing. One is to extend the model to an open economy to allow for international trade. Another possibility is to allow human capital to change endogenously. Yet another direction is to explore quantitative implications of this theoretical model, which presumably requires introducing certain additional relevant frictions to match data.

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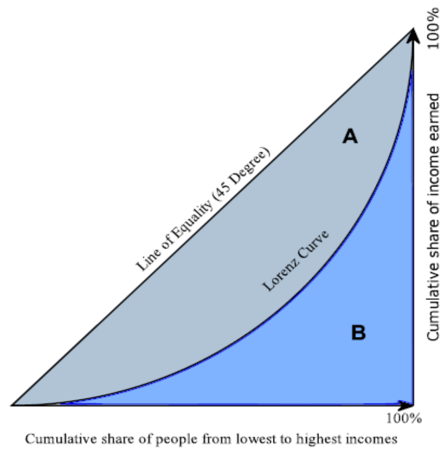
Appendix

Proof of Proposition 1.

Proof. The Gini coefficient is defined mathematically based on the Lorenz curve, which plots the proportion of the total income of the population (y axis) that is cumulatively earned by the bottom x% of the population, see the following diagram:



The line at 45 degrees thus represents perfect equality of incomes. The Gini coefficient can then be thought of as the ratio of the area that lies between the line of equality and the Lorenz curve (marked A in the diagram) over the total area under the line of equality (marked A and B in the diagram); *i.e.*, Gini = A / (A + B).
In Proposition 1, area A is a triangle, the area of which is given by

$$A = \frac{1}{2}\left[(1-\theta) - \frac{(1-\theta)\, L_p W}{(1-\theta)\, L_p W + \theta L_r W}\right] = \frac{1}{2}\,(1-\theta)\,\theta \frac{(L_r - L_p)}{(1-\theta)\, L_p + \theta L_r}$$

and the area of (A + B) is 1/2, so we obtain the Gini coefficient as $(1-\theta)\,\theta \frac{(L_r - L_p)}{(1-\theta)L_p + \theta L_r}$.
Q.E.D.
As a useful rule, the Gini coefficient for this two-group situation is equal to the population proportion of the low-income group minus the income proportion of the low-income group.

Proof of (19)

Proof. Use the rule mentioned in the proof of Proposition 1, the Gini coefficient in the rural region is equal to the population proportion of the low-income group minus the income proportion of the low-income group, that is,

$$\begin{aligned}
& \frac{1-\theta}{\theta\frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}+1-\theta}-\frac{(1-\theta)\,L_pW}{(1-\theta)\,L_pW+\theta\frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}L_rW} \\
= & \frac{1-\theta}{\theta\frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}+1-\theta}-\frac{(1-\theta)\,L_p}{(1-\theta)\,L_p+\theta A_a^{\epsilon-1}A_m^{-\epsilon}} \\
= & \frac{(1-\theta)\,\theta A_a^{\epsilon-1}A_m^{-\epsilon}\left(1-\frac{L_p}{L_r}\right)}{\left(\theta\frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}+1-\theta\right)\left[(1-\theta)\,L_p+\theta A_a^{\epsilon-1}A_m^{-\epsilon}\right]}
\end{aligned}$$

Q.E.D.

Proof of how $GINI_r$ in (34) changes with Ω

Proof. When $\Omega\in[L_p,L_r)$, we have

$$GINI_r=\frac{\theta\,(1-\theta)\left(1-\frac{L_p}{L_r}\right)\Omega}{\left[L_p+\theta\,(\Omega-L_p)\right]\left[1-\theta+\theta\frac{\Omega}{L_r}\right]},$$

which strictly increases with Ω for any $\Omega\in(L_p,\frac{1-\theta}{\theta}\sqrt{L_pL_r})$.

$$L_p<\frac{1-\theta}{\theta}\sqrt{L_pL_r}\Leftrightarrow L_r>\left(\frac{\theta}{1-\theta}\right)^2L_p,$$

which is true if and only if $\theta\leq 1/2$ or $L_r>\left(\frac{\theta}{1-\theta}\right)^2L_p$ when $\theta>1/2$.
Observe that

$$\frac{1-\theta}{\theta}\sqrt{L_pL_r}\leq L_r\Leftrightarrow L_r\geq\left(\frac{1-\theta}{\theta}\right)^2L_p,$$

which is always true if $\theta\geq 1/2$. When $\theta<1/2$, it requires $L_r\geq\left(\frac{1-\theta}{\theta}\right)^2L_p$.
So when $\theta<1/2$, we have $L_p<\frac{1-\theta}{\theta}\sqrt{L_pL_r}$. In this case, if we further have $L_r\geq\left(\frac{1-\theta}{\theta}\right)^2L_p$, then $\frac{1-\theta}{\theta}\sqrt{L_pL_r}\leq L_r$, so $GINI_r$ increases with Ω when $\Omega\in(L_p,\frac{1-\theta}{\theta}\sqrt{L_pL_r})$ and decreases with Ω when $\Omega\in(\frac{1-\theta}{\theta}\sqrt{L_pL_r},L_r]$. If $L_r\leq\left(\frac{1-\theta}{\theta}\right)^2L_p$, then $\frac{1-\theta}{\theta}\sqrt{L_pL_r}\geq L_r$, so $GINI_r$ increases with Ω when $\Omega\in(L_p,L_r]$.
When $\theta>1/2$, we always have $\frac{1-\theta}{\theta}\sqrt{L_pL_r}<L_r$. In this case, $L_p<\frac{1-\theta}{\theta}\sqrt{L_pL_r}$ holds only when $L_r>\left(\frac{\theta}{1-\theta}\right)^2L_p$, so $GINI_r$ increases with Ω when $\Omega\in(L_p,\frac{1-\theta}{\theta}\sqrt{L_pL_r})$ and decreases with Ω when $\Omega\in(\frac{1-\theta}{\theta}\sqrt{L_pL_r},L_r]$. When $L_r\leq\left(\frac{\theta}{1-\theta}\right)^2L_p$, $GINI_r$ decreases with Ω when $\Omega\in(L_p,L_r]$.
When $\theta=1/2$, $L_p<\frac{1-\theta}{\theta}\sqrt{L_pL_r}$ is always true. $GINI_r$ increases with Ω when $\Omega\in(L_p,\frac{1-\theta}{\theta}\sqrt{L_pL_r})$ and decreases with Ω when $\Omega\in(\frac{1-\theta}{\theta}\sqrt{L_pL_r},L_r]$.

When $\Omega=\frac{1-\theta}{\theta}\sqrt{L_pL_r}$, the rural region achieves the maximum inequality with Gini coefficient $GINI_r=\frac{\sqrt{L_r}-\sqrt{L_p}}{\sqrt{L_r}+\sqrt{L_p}}$.
When $\Omega\notin[L_p,L_r)$, we can do the analysis in a similar way. Q.E.D

Proof for How Rural Gini Changes over Time

Proof. Suppose $\theta < \frac{1}{2}$ and (39) is true. Recall Figure 4a refers to the case when $L_r \leq \left(\frac{1-\theta}{\theta}\right)^2 L_p$ and $\theta < \frac{1}{2}$. Since

$$\left[1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right] L_p \leq \left(\frac{1-\theta}{\theta}\right)^2 L_p \Leftrightarrow \theta \leq \frac{1}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 2},$$

so when $\theta \leq \frac{1}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+2}$, Figure 4a applies under (39). So if $\Omega(0) < L^*$, rural gini coefficient monotonically decreases over time. If $\Omega(0) > L^*$, rural gini coefficient monotonically increases over time. When $\theta \in (\frac{1}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+2}, \frac{1}{2})$, we have

$$L_r \geq \left[1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right] L_p > \left(\frac{1-\theta}{\theta}\right)^2 L_p,$$

so Figure 4b applies.
Now we show that $\widetilde{L} \equiv \frac{1-\theta}{\theta}\sqrt{L_p L_r} > L^*$ if and only if

$$1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta} \leq \frac{L_r}{L_p} < H, \tag{47}$$

where H is given by (45):

$$\begin{aligned}
\widetilde{L} &> L^* \Leftrightarrow \frac{1-\theta}{\theta}\sqrt{L_p L_r} > \frac{L_r - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}(\frac{1}{\theta}-1)L_p}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+1} \\
&\Leftrightarrow \frac{L_r}{L_p} - \left(\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1\right)\frac{1-\theta}{\theta}\sqrt{\frac{L_r}{L_p}} - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}(\frac{1}{\theta}-1) < 0 \\
&\Leftrightarrow \sqrt{1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\frac{1}{\theta}} \leq \sqrt{\frac{L_r}{L_p}} < \sqrt{H},
\end{aligned}$$

where the first inequalty in the last line is from (39). We can verify that $1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\frac{1}{\theta} < H$ always holds when $\theta \in (0, \frac{1}{2})$.

So if $\Omega(0) < L^*$, the rural Gini coefficient strictly decrease over time as Ω continuously decreases, till Ω reaches $(1-\theta)L_p$, after which rural Gini coefficient is always zero. If $\Omega(0) \in (L^*, \widetilde{L})$, the rural Gini coefficient first strictly increases till it reaches the maximum value when $\Omega = \widetilde{L}$, after which rural Gini coefficient strictly decreases till $\Omega = L_r$, after which rural Gini coefficient remains constant at the level given by (5). Q.E.D.

Proof of Proposition 9

When the following is true:

$$\theta L_r + L_p(1-\theta) \geq A_a^{\epsilon-1} A_m^{-\epsilon},$$

let

$$T = (1-\theta)\left[\left(\frac{A_a}{A_m}\right)^{\epsilon} - A_a L_p\right]\frac{W}{A_a}$$

$$\begin{aligned}
\theta\left[WL_r - \left(\frac{A_a}{A_m}\right)^{\epsilon}\frac{W}{A_a}\right] - (1-\theta)\left[\left(\frac{A_a}{A_m}\right)^{\epsilon} - A_a L_p\right]\frac{W}{A_a} &> 0 \\
\theta\left[A_a L_r - \left(\frac{A_a}{A_m}\right)^{\epsilon}\right] - (1-\theta)\left[\left(\frac{A_a}{A_m}\right)^{\epsilon} - A_a L_p\right] &> 0
\end{aligned}$$

It can be shown that all poor people can afford to consume $\left(\frac{A_a}{A_m}\right)^{\epsilon}$ amount of agriculture goods and the total output of agriculture good is $\left(\frac{A_a}{A_m}\right)^{\epsilon}$, so the total effective labor to produce agriculture is $A_a^{\epsilon-1}A_m^{-\epsilon}$, and the total expenditure on non-ag is

$$\begin{aligned}
&\theta\left[WL_r - \left(\frac{A_a}{A_m}\right)^{\epsilon}\frac{W}{A_a}\right] - T + T - (1-\theta)\left[\left(\frac{A_a}{A_m}\right)^{\epsilon} - A_a L_p\right]\frac{W}{A_a} \\
=\ & \frac{W}{A_a}\left\{\theta\left[A_a L_r - \left(\frac{A_a}{A_m}\right)^{\epsilon}\right] - (1-\theta)\left[\left(\frac{A_a}{A_m}\right)^{\epsilon} - A_a L_p\right]\right\} \\
=\ & \frac{W}{A_a}\left[\theta A_a L_r + (1-\theta)A_a L_p - \left(\frac{A_a}{A_m}\right)^{\epsilon}\right]
\end{aligned}$$

which is solely produced by rich people. The total GDP is given by

$$\begin{aligned}
&\theta\left[\frac{WL_r - \frac{T}{\theta}}{p_m} + \frac{1}{\epsilon-1}\left(\frac{A_a}{A_m}\right)^{\epsilon-1}\right] + (1-\theta)\left[\frac{WL_p + \frac{T}{1-\theta}}{p_m} + \frac{1}{\epsilon-1}\left(\frac{A_a}{A_m}\right)^{\epsilon-1}\right] \\
=\ & A_m\left[\theta L_r + (1-\theta)L_p\right] + \frac{1}{\epsilon-1}\left(\frac{A_a}{A_m}\right)^{\epsilon-1},
\end{aligned}$$

which is strictly larger than that before the redistribution. Moreover, the non-ag employment share is given by

$$N_m = \theta + (1-\theta)\frac{L_p}{L_r} - \frac{A_a^{\epsilon-1}A_m^{-\epsilon}}{L_r}$$

which is smaller than before.

When

$$\theta L_r + L_p(1-\theta) < A_a^{\epsilon-1}A_m^{-\epsilon} < L_r,$$

we could let

$$T = \theta L_r W - \theta\left(\frac{A_a}{A_m}\right)^{\epsilon}\frac{W}{A_a},$$

and transfer them equally to a subset of poor households with measure equal to

$$\frac{\theta L_r W - \theta\left(\frac{A_a}{A_m}\right)^{\epsilon}\frac{W}{A_a}}{\left[\left(\frac{A_a}{A_m}\right)^{\epsilon} - A_a L_p\right]\frac{W}{A_a}} = \theta\frac{L_r - \left(\frac{A_a}{A_m}\right)^{\epsilon}\frac{1}{A_a}}{\left(\frac{A_a}{A_m}\right)^{\epsilon}\frac{1}{A_a} - L_p} = \theta\frac{L_r - \Omega}{\Omega - L_p},$$

and the remaining poor households with measure equal to

$$1 - \theta - \theta\frac{L_r - \Omega}{\Omega - L_p},$$

will stay with their original consumption. The post-distribution GDP is

$$\begin{aligned}
& \left(\theta + \theta\frac{L_r - \Omega}{\Omega - L_p}\right)\frac{\epsilon}{\epsilon - 1}\left(\frac{A_a}{A_m}\right)^{\epsilon - 1} + \left(1 - \theta - \theta\frac{L_r - \Omega}{\Omega - L_p}\right)\frac{\epsilon}{\epsilon - 1}\left(A_a L_p\right)^{\frac{\epsilon - 1}{\epsilon}} \\
= \; & \theta\left(\frac{L_r - L_p}{\Omega - L_p}\right)\frac{\epsilon}{\epsilon - 1}\left(\frac{A_a}{A_m}\right)^{\epsilon - 1} + \left(1 - \theta - \theta\frac{L_r - \Omega}{\Omega - L_p}\right)\frac{\epsilon}{\epsilon - 1}\left(A_a L_p\right)^{\frac{\epsilon - 1}{\epsilon}} \\
= \; & \frac{\epsilon}{\epsilon - 1}A_m \Omega^{\frac{1}{\epsilon}}\left[\theta\left(\frac{L_r - L_p}{\Omega - L_p}\right)\Omega^{\frac{\epsilon - 1}{\epsilon}} + \left(1 - \theta\left(\frac{L_r - L_p}{\Omega - L_p}\right)\right)L_p^{\frac{\epsilon - 1}{\epsilon}}\right]
\end{aligned}$$

Note that

$$\begin{aligned}
& \theta\left(\frac{L_r - L_p}{\Omega - L_p}\right)\Omega + \left(1 - \theta\left(\frac{L_r - L_p}{\Omega - L_p}\right)\right)L_p \\
= \; & \theta\left(L_r - L_p\right) + L_p
\end{aligned}$$

note that the before-distribution GDP is

$$\frac{\epsilon}{\epsilon - 1}A_m \Omega^{\frac{1}{\epsilon}}\left[\theta\Omega^{-\frac{1}{\epsilon}}\frac{\epsilon - 1}{\epsilon}L_r - \theta\frac{\epsilon - 1}{\epsilon}\Omega^{\frac{\epsilon - 1}{\epsilon}} + \theta\Omega^{\frac{\epsilon - 1}{\epsilon}} + (1 - \theta)L_p^{\frac{\epsilon - 1}{\epsilon}}\right]$$

so

$$\begin{aligned}
\left(\frac{L_r - L_p}{\Omega - L_p}\right)\Omega^{\frac{\epsilon - 1}{\epsilon}} - \left(\frac{L_r - \Omega}{\Omega - L_p}\right)L_p^{\frac{\epsilon - 1}{\epsilon}} & > \Omega^{-\frac{1}{\epsilon}}\frac{\epsilon - 1}{\epsilon}L_r + \frac{1}{\epsilon}\Omega^{\frac{\epsilon - 1}{\epsilon}} \\
\frac{\Omega^{\frac{\epsilon - 1}{\epsilon}} - L_p^{\frac{\epsilon - 1}{\epsilon}}}{\Omega - L_p} & > \Omega^{-\frac{1}{\epsilon}}\frac{\epsilon - 1}{\epsilon},
\end{aligned}$$

which is true because the right hand side is the slope of curve $y = x^{\frac{\epsilon - 1}{\epsilon}}$ at point $x = \Omega$ on the $x - y$ space while the left hand side is larger.

Proof of Proposition 10.

Proof. Consider the following policy: An infinitely high tax rate is imposed on the consumption of agriculture permanently. In this case,

$$Y(t) = A_m(t) \cdot \left[\theta L_r + (1 - \theta)L_p\right],$$

where

$$\dot{A}_m = A_m \left[\theta L_r + (1-\theta) L_p\right]^{\alpha},$$

so aggregate GDP at any time t is given by

$$Y(t) = A_m(0) \cdot \left[\theta L_r + (1-\theta) L_p\right] e^{\left[\theta L_r + (1-\theta) L_p\right]^{\alpha} t},$$

and the total discounted welfare of a rich household is

$$\begin{aligned}
&\int_0^{\infty} \frac{\left[A_m(0) L_r e^{\left[\theta L_r + (1-\theta) L_p\right]^{\alpha} t}\right]^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} e^{-\rho t} dt \\
= &\int_0^{\infty} \frac{\left(A_m(0) L_r\right)^{\left(1-\frac{1}{\sigma}\right)} e^{\left\{\left[\theta L_r + (1-\theta) L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right) - \rho\right\} t} - e^{-\rho t}}{1-\frac{1}{\sigma}} dt \\
= &\frac{1}{1-\frac{1}{\sigma}} \left[\frac{\left(A_m(0) L_r\right)^{1-\frac{1}{\sigma}}}{\rho - \left[\theta L_r + (1-\theta) L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)} - \frac{1}{\rho}\right],
\end{aligned}$$

and the total welfare of a poor household is

$$\frac{1}{1-\frac{1}{\sigma}} \left[\frac{\left(A_m(0) L_p\right)^{1-\frac{1}{\sigma}}}{\rho - \left[\theta L_r + (1-\theta) L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)} - \frac{1}{\rho}\right].$$

Consider another extreme policy, which permanently prohibits production of non-agriculture goods and imposing *a lmup-sum tax* $(1-\theta)(L_r - L_p)W$ *on each rich household and equally redistributing to all poor households in a lump-sum way* every time point, then GDP at time t is given by

$$Y(t) = \frac{\epsilon}{\epsilon - 1} A_a(t)^{\frac{\epsilon-1}{\epsilon}} \left[\theta L_r + (1-\theta) L_p\right]^{\frac{\epsilon-1}{\epsilon}},$$

where

$$\dot{A}_a = A_a \left[\theta L_r + (1-\theta) L_p\right]^{\alpha},$$

so

$$\begin{aligned}
Y(t) &= Y(0) e^{\frac{\epsilon-1}{\epsilon}\left[\theta L_r + (1-\theta) L_p\right]^{\alpha} t} \\
&= \frac{\epsilon}{\epsilon - 1} A_a(0)^{\frac{\epsilon-1}{\epsilon}} \left[\theta L_r + (1-\theta) L_p\right]^{\frac{\epsilon-1}{\epsilon}} e^{\frac{\epsilon-1}{\epsilon}\left[\theta L_r + (1-\theta) L_p\right]^{\alpha} t}
\end{aligned}$$

and the total welfare of a rich household and a poor household will be equal, given by

$$\begin{aligned}
&\int_0^{\infty} \frac{\left[Y(0) e^{\frac{\epsilon-1}{\epsilon}\left[\theta L_r + (1-\theta) L_p\right]^{\alpha} t}\right]^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}} e^{-\rho t} dt \\
= &\int_0^{\infty} \frac{\left(Y(0)\right)^{\left(1-\frac{1}{\sigma}\right)} e^{\left\{\frac{\epsilon-1}{\epsilon}\left[\theta L_r + (1-\theta) L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right) - \rho\right\} t} - e^{-\rho t}}{1-\frac{1}{\sigma}} dt \\
= &\frac{1}{1-\frac{1}{\sigma}} \left[\frac{\left[\frac{\epsilon}{\epsilon-1} A_a(0)^{\frac{\epsilon-1}{\epsilon}} \left[\theta L_r + (1-\theta) L_p\right]^{\frac{\epsilon-1}{\epsilon}}\right]^{1-\frac{1}{\sigma}}}{\rho - \frac{\epsilon-1}{\epsilon}\left[\theta L_r + (1-\theta) L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)} - \frac{1}{\rho}\right],
\end{aligned}$$

Compare the welfare of a rich household in the first policy with the welfare of a household in the second policy

$$\frac{1}{1-\frac{1}{\sigma}}\left[\frac{(A_m(0)L_r)^{1-\frac{1}{\sigma}}}{\rho-\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}-\frac{1}{\rho}\right]-\frac{1}{1-\frac{1}{\sigma}}\left[\frac{\left[\frac{\epsilon}{\epsilon-1}A_a(0)^{\frac{\epsilon-1}{\epsilon}}\left[\theta L_r+(1-\theta)\,L_p\right]^{\frac{\epsilon-1}{\epsilon}}\right]^{1-\frac{1}{\sigma}}}{\rho-\frac{\epsilon-1}{\epsilon}\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}-\frac{1}{\rho}\right]$$

$$=\frac{1}{1-\frac{1}{\sigma}}\left[\frac{(A_m(0)L_r)^{1-\frac{1}{\sigma}}}{\rho-\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}-\frac{\left[\frac{\epsilon}{\epsilon-1}A_a(0)^{\frac{\epsilon-1}{\epsilon}}\left[\theta L_r+(1-\theta)\,L_p\right]^{\frac{\epsilon-1}{\epsilon}}\right]^{1-\frac{1}{\sigma}}}{\rho-\frac{\epsilon-1}{\epsilon}\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}\right]$$

We can show that it is strictly postivie if and only if when $\Omega(0)<\Omega_1$, a poor household's welfare in the first policy is higher than that of a household in the second policy if and only if $\Omega(0)<\Omega_2$, where $\Omega_2<\Omega_1$.
When $\Omega(0)<\Omega_2$, every household is strictly better off under Policy A than under Policy B. When $\Omega(0)\in(\Omega_2,\,\Omega_1)$, every rich household is strictly better off under Policy A than under Policy B but the opposite is true for each poor household. When $\Omega(0)>\Omega_1$, every household is strictly worse off under Policy A than under Policy B. When $\Omega(0)=\Omega_2$, every poor household feels indifferent between the two policies but every rich household strictly prefers Policy A. When $\Omega(0)=\Omega_1$, every rich household feels indifferent between the two policies but every poor household strictly prefers Policy B.

$$\Omega_1 \equiv \left[\frac{\rho-\frac{\epsilon-1}{\epsilon}\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}{\rho-\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}\right]^{\frac{\sigma\epsilon}{\sigma-1}}\left(\frac{L_r}{\frac{\epsilon}{\epsilon-1}\left[\theta L_r+(1-\theta)\,L_p\right]^{\frac{\epsilon-1}{\epsilon}}}\right)^{\epsilon}$$

$$\Omega_2 \equiv \left[\frac{\rho-\frac{\epsilon-1}{\epsilon}\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}{\rho-\left[\theta L_r+(1-\theta)\,L_p\right]^{\alpha}\left(1-\frac{1}{\sigma}\right)}\right]^{\frac{\sigma\epsilon}{\sigma-1}}\left(\frac{L_p}{\frac{\epsilon}{\epsilon-1}\left[\theta L_r+(1-\theta)\,L_p\right]^{\frac{\epsilon-1}{\epsilon}}}\right)^{\epsilon}$$

By comparison, the Laissez-affaire market equilibrium in the steady state described in Proposition 5, the welfare of a rich household is given by

$$c_r=A_mL_r+\frac{1}{\epsilon-1}\left(\frac{A_a}{A_m}\right)^{\epsilon-1}=A_m\left[L_r+\frac{1}{\epsilon-1}\Omega\right],$$

$$g_{GDP}=\frac{\dot{A}_m}{A_m}=\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1}\left[\theta L_r+(1-\theta)L_p\right]\right]^{\alpha}$$

$$\int_0^\infty \frac{\left[A_m(0)\left[L_r+\frac{1}{\epsilon-1}\Omega\right]e^{\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1}[\theta L_r+(1-\theta)L_p]\right]^\alpha t}\right]^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}e^{-\rho t}dt$$

$$= \int_0^\infty \frac{\left(A_m(0)\left[L_r+\frac{1}{\epsilon-1}\Omega\right]\right)^{\left(1-\frac{1}{\sigma}\right)}e^{\left\{\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1}[\theta L_r+(1-\theta)L_p]\right]^\alpha\left(1-\frac{1}{\sigma}\right)-\rho\right\}t}-e^{-\rho t}}{1-\frac{1}{\sigma}}dt$$

$$= \frac{1}{1-\frac{1}{\sigma}}\left[\frac{\left(A_m(0)\left[L_r+\frac{1}{\epsilon-1}\Omega^*\right]\right)^{1-\frac{1}{\sigma}}}{\rho-\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1}\left[\theta L_r+(1-\theta)L_p\right]\right]^\alpha\left(1-\frac{1}{\sigma}\right)}-\frac{1}{\rho}\right],$$

and the total welfare of a poor household is

$$c_r=\frac{\epsilon}{\epsilon-1}\left(A_aL_p\right)^{\frac{\epsilon-1}{\epsilon}}=A_m\frac{\epsilon}{\epsilon-1}L_p^{\frac{\epsilon-1}{\epsilon}}\Omega^{\frac{1}{\epsilon}}$$

the total welfare is

$$\frac{1}{1-\frac{1}{\sigma}}\left[\frac{\left(A_m(0)\frac{\epsilon}{\epsilon-1}L_p^{\frac{\epsilon-1}{\epsilon}}\Omega^{\frac{1}{\epsilon}}\right)^{1-\frac{1}{\sigma}}}{\rho-\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1}\left[\theta L_r+(1-\theta)L_p\right]\right]^\alpha\left(1-\frac{1}{\sigma}\right)}-\frac{1}{\rho}\right].$$

A rich household is strictly better off in policy 1 than in Laissez-faire steady state in proposition 5 if and only if

$$\frac{1}{1-\frac{1}{\sigma}}\left[\frac{\left(A_m(0)L_r\right)^{1-\frac{1}{\sigma}}}{\rho-\left[\theta L_r+(1-\theta)L_p\right]^\alpha\left(1-\frac{1}{\sigma}\right)}-\frac{\left(A_m(0)\left[L_r+\frac{1}{\epsilon-1}\frac{L_r-\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\left(\frac{1}{\theta}-1\right)L_p}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+1}\right]\right)^{1-\frac{1}{\sigma}}}{\rho-\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1}\left[\theta L_r+(1-\theta)L_p\right]\right]^\alpha\left(1-\frac{1}{\sigma}\right)}\right]>0.$$

When $\frac{1}{1-\frac{1}{\sigma}}>0$, it becomes

$$\frac{1}{\rho-\left[\theta L_r+(1-\theta)L_p\right]^\alpha\left(1-\frac{1}{\sigma}\right)}>\frac{\left(\left[1+\frac{1}{\epsilon-1}\frac{1-\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}\left(\frac{1}{\theta}-1\right)\frac{L_p}{L_r}}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}}+1}\right]\right)^{1-\frac{1}{\sigma}}}{\rho-\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}+1}\left[\theta L_r+(1-\theta)L_p\right]\right]^\alpha\left(1-\frac{1}{\sigma}\right)},$$

which is never possible.
When $\frac{1}{1-\frac{1}{\sigma}} < 0$, it becomes

$$\frac{1}{\rho + \left[\theta L_r + (1-\theta) L_p\right]^{\alpha} \left(\frac{1}{\sigma} - 1\right)} < \frac{\left(\left[1 + \frac{1}{\epsilon - 1} \frac{1 - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \left(\frac{1}{\theta} - 1\right) \frac{L_p}{L_r}}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1}\right]\right)^{1-\frac{1}{\sigma}}}{\rho + \left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1} \left[\theta L_r + (1-\theta) L_p\right]\right]^{\alpha} \left(\frac{1}{\sigma} - 1\right)},$$

which is reduced to

$$\rho < \frac{\left[\theta L_r + (1-\theta) L_p\right]^{\alpha} \left(\frac{1}{\sigma} - 1\right) \left[\left[1 + \frac{1}{\epsilon - 1} \frac{1 - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \left(\frac{1}{\theta} - 1\right) \frac{L_p}{L_r}}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1}\right]^{1-\frac{1}{\sigma}} - \left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1}\right]^{\sigma}\right]}{1 - \left[1 + \frac{1}{\epsilon - 1} \frac{1 - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \left(\frac{1}{\theta} - 1\right) \frac{L_p}{L_r}}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1}\right]^{1-\frac{1}{\sigma}}},$$

which only requires that

$$\left[1 + \frac{1}{\epsilon - 1} \frac{1 - \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \left(\frac{1}{\theta} - 1\right) \frac{L_p}{L_r}}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1}\right]^{\frac{\sigma-1}{\sigma}} > \left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1}\right]^{\sigma},$$

which can be further simplified to

$$\frac{1 - (\epsilon - 1)\left(\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1\right) \left[\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1}\right]^{\frac{\sigma^2}{\sigma-1}} - 1\right]}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \left(\frac{1}{\theta} - 1\right)} < \frac{L_p}{L_r}.$$

Together with (39), we must have

$$\frac{1}{\left[1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right]} \geq \frac{L_p}{L_r} > \frac{1 - (\epsilon - 1)\left(\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} + 1\right) \left[\left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1}\right]^{\frac{\sigma^2}{\sigma-1}} - 1\right]}{\left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \left(\frac{1}{\theta} - 1\right)},$$

which further requires

$$\frac{\log \frac{1 + \left(1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right)(\epsilon - 1)}{\left(1 + \left[\frac{\epsilon-1}{\epsilon}\right]^{\frac{1}{\alpha}} \frac{1}{\theta}\right)(\epsilon - 1)}}{\log \left[\frac{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}} + 1}{\left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1}{\alpha}}}\right]} < \frac{\sigma^2}{1 - \sigma},$$

since the left hand side is strictly positive and right hand side is a strictly increasing function of σ for $\sigma \in [0, 1)$ with value from 0 to ∞, so there must exist a unique $\sigma^* \in (0, 1)$ such that the above inequality is true if and only if $\sigma \in (\sigma^*, 1)$. Q.E.D.