The Cyclical Behavior of Factor Shares

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Abstract

We review the empirical evidence about factor shares and show that, apart from a varying trend, they are characterized by a strong and persistent cyclical pattern. A typical expansion begins with an increase in the capital income share; this share peaks substantially earlier than output, and falls in the the last phase of expansion. Next, we provide a theory of why this may be due to the pattern of technological innovation under competition. Central to our theory are endogenous movements in relative factor prices creating incentives for replacing old technologies with new ones. Accumulation of capital increases the labor share in the short run; in the longer run, a rising labor cost incentivizes firms to innovate on labor saving technologies, the adoption of which eventually reduces the labor share. This endogenous interaction between labor-saving innovations and changes in the relative price of labor is the source of both growth and cycles.

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1 Introduction

In this paper, we review the empirical evidence about factor shares and show that, apart from a varying trend[1], they are characterized by a strong and persistent cyclical pattern. A typical expansion begins with an increase in the capital income share; this share peaks substantially earlier than output; and falls in the last phase of the expansion. Our paper investigates the role that endogenous decentralized technological change plays in determining growth and cycles, through the lens of factor income shares. In our theory, factor prices change over time due to the accumulation of productive capacity; such changes create incentives for variations in the rate of accumulation of productive capacity, the introduction of new technologies and its replacement with old ones. Put differently, we study endogenous technological progress that is 'biased' by the relative price of inputs, and derive a model in which persistent growth, persistent business fluctuations, and persistent movements in the share of income going to, respectively, labor and capital are simultaneously determined.

We focus on decentralized technological change: neither aggregate technological progress nor aggregate productivity or preference shocks are assumed. All decisions, crucially the adoption of new methods of production, take place at the firm's level. They come about as profit maximizing responses to changes in relative prices and in the equilibrium conditions of competitive markets. Heterogeneous technologies, as opposed to an aggregate production set, are the elementary units of analysis. Aggregate and oscillatory growth is shown to persist even when exogenous sources of uncertainty are set to zero in a fully deterministic setting.

In the model, there is a countably infinite vintages of capital goods. Later capital embody more labor saving technology in the sense that it requires less labor input to produce one unit of output. Growth in total factor productivity is endogenous and results from the adoption of new technologies – capital deepening – their subsequent expansion – capital widening – and their eventual replacement with better ones – capital scrapping. The duration of each phase is endogenous, and determined by the equilibrium movements in the relative prices of labor and (different kinds of) capital. Recessions occur when capital widening has reached its upper limit and scrapping, followed by deepening, becomes economically beneficial; expansions set in when capital deepening is successful and widening may be undertaken at a higher than normal rate.[2]

Apart from the goal of building a theoretical model in which growth and cycles are joint equilibrium outcomes, our motivations are also empirical. For a theory of this kind to stand the chance of turning into a quantitative model of actual growth and cycles, its equi-

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[1]See discussion in recent literature, e.g. Karabarbounis and Neiman (2014).

[2]Recessions in our model are 'caused' by innovation; and labor, rather than credit or capital, is the factor that matters. For a story that centers around credit capacity: credit is cheap at beginning of expansion, hence firms take a lot of credit, credit becomes scarce eventually and it price goes up, killing lots of bad projects, those that use more credit than average, and have lower value added, and inducing recession, the challenge seems in how to 'create' new credit capacity after the recession cleans up bad projects.

librium paths must display a fairly long list of qualitative features. A relatively stable long run trend should obtain, around which cycles of varying length, between three and ten years, are observed. Quarterly growth rates in output are positive most of the time, but negative growth may occur at a fairly infrequent rate. Growth rates are positively auto-correlated. Income, consumption, investment, and labor productivity are co-integrated series. Positive TFP obtains when quality adjustments in capital and labor are not made. Consumption and investment are pro-cyclical, but the latter oscillates more than output while the former substantially less. Productivity of labor is mostly pro-cyclical while real wages are only weakly so. Factor shares follow a cyclical pattern, with delays: the share of capital is pro-cyclical but peaks before total output does while the labor share is countercyclical but bottoms out after the recession ends[3].

While most of these stylized facts are extensively documented in the literature (e.g. Cooley and Prescott, 1995; King and Rebelo, 1999), some are worthy of a few additional words. First, the shares of income accruing to capital and labor move in a systematic way with the business cycle. Second, profits and the growth rate of labor productivity, beside being correlated, are pro-cyclical, but peak substantially earlier than the cycle does; that is, when recessions set in, profits have been decreasing and labor productivity has stopped growing for a few quarters already. Third, while there is very little short term substitutability between capital and labor, the substitutability is substantial in the longer run, and most technological improvements appear to be labor saving. Fourth: empirical evidence suggests that employment drops on impact when a permanent technological improvement arrives, and start rising again only quite a few quarters later.

Standard business cycle models have a difficult time explaining one or more of these facts, even when assuming that growth is exogenous and that aggregate and autocorrelated exogenous technology and preference shocks are the main driving forces. That this is the case for any model working with an aggregate Cobb-Douglas production function, should be obvious; retaining a Cobb-Douglas production function and making the share parameter stochastic (Young, 2004; Rios-Rull and Santaeulalia-LIopis, 2010), beyond generating a counterfactually high correlation between capital share and output, is dangerously close to assuming a trivial answer (exogenous movements) to the empirical puzzle. Less obvious is the fact that the easy fix, a CES production function with an elasticity of substitution different from one, is actually not a fix. If we compute a standard RBC model with CES production function and an elasticity of substitution similar to the ones reported in the literature (Antràs, 2004; Oberfield and Raval, 2014), we find that such models predict movements in factor shares quite smaller than the observed ones[4]. Moreover, if we assume that technological progress is Harrod-neutral, as required to have a stationary capital-output ratio in models of exogenous growth, wages become strongly countercyclical, contrary to empirical evidence.

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[3]See also Rios-Rull and Santaeulalia-LIopis (2010).

[4]The estimated elasticity of substitution lies in the range 0.6 to 0.9 in Antràs (2004), and about 0.7 for the manufacturing sector in Oberfield and Raval (2014). We have shown in appendix that under such an elasticity of substitution, technology shock with a CES production function would generate variation in factor income share that is too small comparing to data.

Models with sticky prices and/or sticky wages do not have an easier time at capturing the facts. In response to a monetary shock, wages will go up because of a higher demand for labor, and labor productivity will go down as labor increases faster than capital, hence the capital income share will go down during an expansion driven by a positive monetary shock. Only in the, empirically unlikely, event that nominal wages are completely rigid and prices adjust very rapidly to the monetary shock, would real wages decrease. Only in the, even more unlikely, event that real wages decrease more than labor productivity does as demand for labor increases, will profit display a pro-cyclical tendency. Leaving aside the fact that this does not seem to have ever happened in the business cycles of the real world, to achieve this we would need wage rigidity to last many quarters in the face of continuous monetary supply surprises and raising prices, an improbability to say the least. The same argument applies, without the latter caveat, if the expansion is driven by some Non-Ricardian fiscal 'stimulus'.

Focusing upon the cyclical movements in factor shares, labor productivity, wages and profit rates, beyond clarifying the problems of existing business cycle models, also tells us the element that a successful theory requires: a mechanism to increase labor productivity faster than wages at the beginning and slower at the end of the expansion. Our paper focuses on one possible channel: the endogenous adoption of labor saving technology by competitive firms. At the beginning of the expansion, firms will pick new technologies that are labor-saving relatively to previous ones. As the latter are scrapped and the new capital that embodies the more efficient technology is accumulated, labor moves accordingly and its productivity increases faster than wages, hence the capital share and output increase rapidly. However, as the replacement process completes and more and more labor is employed, wages will eventually go up, drying the corporate profits, reducing investment, and finishing with it the expansion. Only at the bottom of the recession, after old and inefficient productive capacity has been scrapped, a new technology is introduced and the whole cycle starts again.

We are not the first to deal with some of the issues discussed above. To the best of our knowledge, though, we are the first to make the claim that a sound theory of endogenous growth and cycles can be built upon the observation that firms expand productive capacity when they expect the adopted technology to yield a profit in future periods, while they reduce capacity and try to change their technology when they realize the latter is no longer profitable at the expected equilibrium prices. Let us leave aside the very vast literature concerned with endogenous growth and cycles; it suffices here to say that nowhere in that literature one can find a model in which both growth and cycles obtain[5]. We will also spare the reader a long survey of the century-long debate on the nature of technological progress, its biased-ness in one direction or another and the extent to which Harrod-neutral exogenous productivity does or does not mimic the data in a satisfactory

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[5]Explicit mention should be made, though, of Goodwin (1968) and Reichlin (1986). The economic intuition underlying the endogenous oscillations in these two models is quite akin to ours. However, technological innovation being absent, there is no growth, either exogenous or endogenous, in either model.

form. To us, that technological progress must be labor –more generally natural resource– saving is almost tautological beside being blatantly evident. The relevant issues are how to best model this fact, and if the pace at which technological change advances should or should not be made responsive to movements in factor prices.

There are at least three branches of literature interested in the evolution of the factor income shares and the business cycle. First, there have been papers that focused on the distribution of risk over the cycle. Boldrin and Horvath (1995) present a real business cycle model of contractual arrangements between employees and employers where the former are prevented from accessing capital markets and are more risk-adverse than the latter. The paper characterizes an optimal contract that maps the aggregate states of the economy into wages and labor market outcomes. The optimal contract, which provides insurance for workers, prevents decline of wage in a negative productivity shock in recession and generates a countercyclical labor share. Similarly, Gomme and Greenwood (1995) build a model where workers purchase insurance from the entrepreneurs through optimal contracts[6]. Since our model assumes complete markets, none of the considerations used in those papers is directly pertinent to the mechanism explored here, even if, the introduction of risk-sharing contractual arrangements would reinforce some of the conclusions.

The second branch of the literature has focused on explanations based on models with imperfect competition and/or increasing returns to scale. Hornstein (1993) developed a model of monopolistic competition where the capital income share is pro-cyclical. However, the correlation between output and capital share is perfect, hence the cyclical 'hump-shape' pattern for profits cannot be replicated. Other examples include Ambler and Cardia (1998), Bils (1987), and the models surveyed in the Rotemberg and Woodford (1999). Hansen and Prescott (2005) is an additional contribution along the same lines, which does not make use of monopolistic competition but, instead, of fixed capacity at the plant level.

Third, and the most relevant for us, is the literature spearheaded by Blanchard et al. (1997) and Caballero and Hammour (1998). These papers have explored the dynamics over the middle-run induced by exogenous changes in real wages. After an initial increase in wages, due for example to an exogenous strengthening of the bargaining power of workers, the capital share goes down. What happens over time depends on the long-run elasticity of substitution, either with a permanent fall on capital share or with a return to the initial level. Blanchard et al. (1997) suggests that changes in efficiency induced by the original increase in wages may even increase the long-run share of capital income. Some of the intuitive arguments given by Blanchard, inspired by the European experience in the 1970s and 1980s, are close in spirit to the model we suggest here, in particular the idea that, facing a persistently high exogenous wage, firms may strive to adopt technologies that reduce the labor input per unit of output, thereby leading to an eventual decrease in

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[6]In business cycle models with frictional labor markets (Merz, 1995; Andolfatto, 1996), noncompetitive wage setting through Nash bargaining also generates a gap between wage and labor productivity and a countercyclical labor share.

the share of labor income[7].

The key difference in our investigation is that we do not begin with an initial, exogenously given shock to wages (due to a change in technology, bargaining power or mark-up) and explore the aggregate dynamics after such shock. Further, we view the changes in capital income share as a systematic and recurrent feature of the economy: the main driving force behind the introduction of new technologies and, therefore, of sustained growth. To put it plainly, we posit that growth comes through oscillations in the rate of technology adoption, that such oscillations are endogenous, and that their main source is the explicit conflict over the shares of income going to different factors.

Finally, we note the similarities between some points of our model and the literature on directed technological change surveyed by Acemoglu (2002). Three macroscopic differences are that (i) we claim business cycles are 'caused' by labor-saving technological change, (ii) we focus on the fundamental bias (labor vs capital) in a perfectly competitive environment, and, (iii) we make the bias endogenous and not exogenous. In a recent paper, Acemoglu and Restrepo (2018) endogenizes the direction of innovation, automation or creation of new labor intensive tasks, and allows it to be responsive to factor prices. A temporary shock in automation reduces the labor share in the short run, but also induce R&D efforts in creating new labor intensive tasks, which stabilize factor shares in the long run. Different from that paper, there is no balanced growth path in our model, the economy endogenously alternates between a phase when labor share declines and one with a rising labor income share.

The rest of paper is organized as following: Section 2 presents the stylized facts. Section 3 outlines the basic model and characterizes the competitive equilibrium, where we define the planner's problem and uses its properties to provide further insights into the dynamics predicted by our theory. A discussion on de-trending and medium-term cycles is provided in Section 4. Section 5 concludes.

2   Stylized Facts

In this section we discuss some of the U.S. evidence pertinent to the variation of the capital income share over the business cycle. As shown below, an expansion typically begins with increases in the capital share; this share peaks substantially earlier than the expansion in output, and falls in the last phase of expansion. The capital income share not only is cyclical, that is it reaches local minimum in recessions; it also shows a rise-and-then-fall hump-shaped pattern during an expansion[8].

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[7]The recent empirical research, e.g. Acemoglu and Pascual (2018), that document an increasing adoption of robots when firms face rising labor costs due to demographic change, also give a similar intuition.

[8]Abundant evidence on the cyclical profit share is also available for pretty much each and every EU country, see OECD (2015), ILO (2019). The stylized facts reported here are even more clearly visible in the European post-WWII data, which is what motivated Blanchard and Caballero-Hammour initial work.

We compute the capital share of the U.S. economy in three different ways. First, we compute the capital share for the whole economy. This measure has the advantage of comprehensiveness but the drawback that it includes the household and government sectors whose output is not sold in the market and which have a fixed capital share by construction. Moreover, we need to handle the distribution of proprietor's income between (imputed) wages and capital income. To overcome some of these difficulties, we compute the capital share for the corporate sector. Finally, we compute the capital share for the non-financial corporate sector.

Our first take at evaluating the capital share in the U.S. economy uses aggregate data from the whole economy. As explained before, following this route faces the basic difficulty of how to divide proprietor's income between labor and capital. A common solution is to split proprietors income according to the share of capital income observed in the rest of economy (Cooley and Prescott, 1995). To do so, we can subtract from our measure of output the proprietor's income and include as capital income only the unambiguous capital income.

Capital income includes income coming from two different sources: (1) unambiguous capital income, equal to *rental income of persons*, *corporate profits*, and *net interest and miscellaneous payments*; and (2) the consumption of fixed capital by the non-proprietors private sector and the government. We define output as Gross Nation Product less *proprietors income*. In addition, we subtract the statistical discrepancy between Net National Product and National Income and *net taxes on production and imports*, since both items cannot be divided between capital and labor[9]. As a consequence, the gross capital share is defined as

$$KS = \frac{\text{unambiguous capital income} + \text{depreciation}}{\text{national inc.} + \text{depreciation} - \text{proprietors income} - \text{taxes on P\&I}},$$

and the net capital share is defined as

$$KS^{net} = \frac{\text{unambiguous capital income}}{\text{national inc.} - \text{proprietors income} - \text{taxes on P\&I}}.$$

Figure 6.2 plots the gross capital income share in the whole economy. Clearly, the capital share fluctuates quite a bit. Also plotted is a H-P trend with $\lambda = 1600$ and the NBER dating of recessions. The capital share tends to go up at the beginning of the expansion, peaks at the middle, and drops in the second half of the expansion. The aggregate capital income share can be decomposed into two parts, net capital income and depreciation. Figure 6.3 plots the non-depreciation component of gross capital income share. It is clear that the *corporate profits* component largely accounts for the cyclical pattern of the overall

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Figure 6.12 in appendix shows a clearly negative correlation between the deviation of labor share and value added in the manufacturing sector in France.

[9]Data for our measures are taken directly from NIPA, Table 1.7.5 'Relation of gross domestic product, gross national product, net national product, national income, and personal income', and Table 1.12 'National income by type of income'. Since we only need percentages, we take nominal quantities that avoid distortions induced by price indexes. Our sample, of quarterly data, goes from 1947-q2 to 2018-q2.

capital share. Net interest is relatively a-cyclical[10]. The rental income is also relatively smooth over time. Figure 6.6 shows that the cyclical pattern is not affected if we focus on net capital income shares[11].

Our next measure of the capital share uses data from the corporate sector. This measure is closer to the main theoretical thrust of the paper. We define the output of the corporate sector to be equal to the gross value added of corporate business sector less the Taxes on production and imports net of subsidies. As capital income we add the *net operating surplus* plus *consumption of fixed capital*. We repeat the exercise with the same concepts for the Non-financial corporate sector[12]. Figures 6.7 and 6.8 plot respectively the gross and net capital income share in the corporate business sector. The capital income share fluctuates over business cycles as that in the whole economy. Figures 6.9 and 6.10 confirm the cyclical fluctuation of capital income shares in the non-financial corporate business sector.

The cyclical pattern becomes even clearer if we focus the three longest expansions, that in the 1960s, 1980s, and 1990s, in the United States after WW-II. These three episodes are particularly interesting because the length of expansion allows one to identify more clearly the type of phenomena we are concerned with. Table 2.1 reports the evolution of *corporate profits* in value added in the non-financial corporate business sector[13]. We observe a common structure: corporate profits in particular, and capital income in general, go up at the beginning of the expansion by a considerable amount, peaks roughly at the middle of the expansion, and decreases in the last phase.

Table 2.1: Fraction of corporate profits in value added

	Initial Value	Max. Value	Increase	Final Value	Decrease
Expansion 60s	17.6%	22.4%	27.0%	15.4%	44.9%
Expansion 80s	9.6%	13.7%	43.0%	11.5%	18.8%
Expansion 90s	11.9%	16.9%	42.2%	10.9%	54.6%

Note: values are for the non-financial corporate business sector

The main message of this table is the sizable changes in the *corporate profits* over the business cycle, especially for the 80s and 90s cycles. For example, in the 80s, corporate profits went up by a 43% and in the 90s by a 42.2%[14]. If take a benchmark capital-output ratio of

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[10]The big increase of net interest income in the 1980's is associated with a high real interest rate during that time

[11]Koh et al. (2018) documents the importance of Intellectual property products (IPP) for understanding the long run trend of labor share. As shown in that paper, the cyclical pattern of labor share is not affected by IPP adjustment.

[12]The measures are taken directly from NIPA, Table 1.14. ('Gross value added of domestic corporate business in current dollars and gross value added of nonfinancial domestic corporate business in current and chained dollars'). As before, we employ nominal quantities.

[13]See Table 6.1 in appendix for the fraction of *net operating surplus* in value added in the non-financial corporate business sector.

[14]While this is not the topic of the present paper, one may want to consider how much these dramatic

3, the profitability rate of the corporate sector went in the 90's from 4% to 5.6% and then fell again to 3.6%.

On the other hand, depreciation, or consumption of fixed capital, demonstrates a procyclical pattern, as shown in Figure 6.11. *Consumption of fixed capital* measures the decline, during the course of the accounting period, in the *current* value of the stock of fixed assets owned by a producer as a result of physical deterioration, normal obsolescence or normal accidental damage[15]. It therefore contains both physical and economic depreciation. Depreciation peaks in recessions; the correlation coefficient between the cyclical component of depreciation and that of net operating surplus is -0.7.

According to trough and peaks of the gross capital income share in the non-financial corporate business sector[16], we divide all quarters into two subsets: (1), *KSIN*, quarters when the capital share increases from a trough to the following peak; and (2), *KSDE* when the capital share decreases from a peak to next trough. Table 2.2 presents the avegrage growth rate, i.e. percentage change from previous quarter at annual rate, of 5 variables in these two sub-periods: real gross value added, labor productivity, working hours, employment, and real hourly compensation.

Table 2.2: Moments in increasing *vs* decreasing KSTr periods

growth rate	Mean	Std. Dev.	Mean	Std. Dev.
		KSIN		*KSDE*
value added	5.12	6.79	2.88	5.72
labor productivity	3.42	4.49	1.25	3.47
consumption	3.77	3.32	2.96	3.27
employment	1.45	4.07	2.10	3.17
working hours	1.65	5.03	1.57	3.67
real wage	1.00	3.75	1.71	3.06

Note: growth rate is percent change from previous quarter at annual rate for the non-financial corporate sector from 1947q2 to 2018q2. In total, there are 136 quarters with capital share going up, and 149 quarters with capital share going down. Consumption denotes *real personal consumption expenditure*.

In periods when the capital income share increases, typically the first phase of an expansion, labor productivity and output grow at a rapid rate; and employment and real wage increases relatively slow. Together these lead to a decreasing labor income share. On the other hand, the labor income share rises in the second phase of expansion as labor productivity slows down and wage picks up. Though consumption is expected to be relatively smooth, the average growth rate of consumption in the first phase of expansion is

increases in profitability account for the large stock market rallies witnessed during those two expansions, and for the early 1970*s* and 2000 crashes as well.

[15]This definition is from the 2008 manual of the United Nations System of National Accounts.

[16]Peak quarters are: 1950q3, 1955q1, 1959q2, 1965q2, 1972q4, 1980q2, 1982q4, 1992q4, 2000q4 ,and 2008q1; and trough quarters are: 1953q4, 1958q1, 1961q1, 1970q1, 1974q3, 1981q3, 1984q2, 1997q3, 2006q3, and 2014q3.

still larger than the second phase. There is no significant difference in terms of average working hours across two phases[17].

3   The Model

We introduce here the basic framework under the assumptions of a representative agent and of recursively complete financial markets. In the model, there is an infinite vintage of capital goods, which each is combined with labor to produce the final consumption goods. Technologies are embodied in capital goods, and later capital embodies more advanced and labor saving technology in the sense that it produces one unit of final goods with less labor. Capital goods can self-accumulate, and can also be used to innovate and produce capital goods of the next generation. We study the implication of this endogenous process of accumulation and innovation on factor shares, growth and cycles.

Preferences   We start with the case of exogenous labor supply, and introduce endogenous labor supply later. In the exogenous case, the representative household maximizes the following expected utility over the infinite horizon,

$$\max E_t \int_0^\infty e^{-\rho t} \log c(t) dt.$$

Production   Production takes places in three different sectors denoted by $s = 1, 2, 3$. Each sector is composed by a continuum of size $k^s(t)$ of identical firms[18]. The measures $k^s(t)$ evolve endogenously over time, as detailed below. The first sector produces consumption goods, the second capital goods and the third new technologies embodied in new kinds of capital goods, as detailed below.

Technologies   There exists a countable number of technologies, indexed by the subscript $j = 0, 1, ...$ Technologies are embodied in capital goods, hence $k_j^s(t)$ is the stock of capital j[19] installed in sector s at time t. We say that a technology j is active in sector s during period t if $k_j^s(t) > 0$[20].

Technological Progress   A technology with an index j is better than a technology with index $j' < j$ for two reasons. First, to produce one unit of final consumption, a unit of capital of type j requires less labor than a unit of capital of type j', i.e. technological progress is labor saving. Secondly, technological progress is incremental insofar as capital goods of type $j + 1$ can be obtained, at a cost, only from capital goods of type j and not from any

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[17]We have also calculated the average growth of return to capital in these two sub-periods. The capital return data is taken from Gomme, Ravikumar, and Rupert (2011). The average change in capital return is 0.31 during the *KSUP* phase, and -0.15 for the *KSDN* phase.

[18]Because firms are identical in each sector, we will talk, indifferently, either of a representative firm with a stock of capital equal to $k^s(t)$ or of a measure $k^s(t)$ of identical firms, each one with a unit of capital.

[19]We use capital j to denote productive capacity of vintage j.

[20]Think of technologies as plants, with constant returns to scale.

other $j' < j$.

Consumption Sector   The first sector produces aggregate consumption, $c(t)$, using type j capital $k_j^1(t)$ and labor, $\ell(t)$ according to a fixed coefficient production function,

$$c(t) = \min\{k_j^1(t), \gamma^j \ell(t)\}, \quad \gamma > 1.$$

Capital and labor are assumed to be complementary goods in production.[21] Assume that the labor augmenting coefficient γ is greater than 1. That is, technical progress is labor saving. More advanced technology, i.e. capital with a larger index j, requires less labor ($\frac{1}{\gamma^j}$) to produce 1 unit of final goods.

Capital Widening Sector   The second sector produces additional capital of type j from capital of the same type according to the widening equation,

$$\dot{k}_j(t) = b k_j^2(t),$$

with $b > 0$. The widening technology allows capital to self-accumulate at the rate b.

Capital Deepening Sector   The third sector produces a new type of capital of type $j+1$ from capital of type j according to the deepening equation,

$$k_{j+1}(t) = \frac{k_j^3(t)}{a},$$

with $a > 1$. Capital used in the deepening sector fully depreciates. Capital $j+1$ can only be obtained from capital j, but not directly from any $j', j' < j$. However, capital $j+1$ can be converted from capital $j', j' < j$ by applying the innovation technology $j+1-j'$ times.

At any point in time t the following resource constraint holds for capital of vintage j,

$$k_j(t) = k_j^1(t) + k_j^2(t) + k_j^3(t).$$

That is, capital j can be employed in either of the three sectors. The *accumulation equation* for capital j is,

$$dk_j(t) = b k_j^2(t) dt - k_j^3(t) + \frac{k_{j-1}^3(t)}{a}.$$

The stock of capital j changes due to, self accumulation as in the first term, depreciation if used in innovating on capital $j+1$, or innovation from capital $j-1$ in the capital widening sector. Note that we allow for discrete conversion of quality j capital flow to quality $j+1$.

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[21]See the empirical literature, e.g. Antras (2004), Oberfield and Raval (2014), which estimate the elasticity of substitution between capital and labor. We employ a Leontief production function to deliver analytical solutions. All qualitative results hold for a production function with gross capital-labor complementarity.

The key technical property is that this economy is an ordinary diminishing return economy with three sectors: consumption, widening and deepening. Diminishing return to capital accumulation derives from the fact that capital and labor are complementary inputs and labor supply is upper bounded. This motivation of innovation is therefore different from creative destruction models (Aghion and Howitt, 1992; Grossman and Helpman, 1991) where firms innovate to obtain monopoly power. Note that as there is perfect competition, the second welfare theorem holds. The efficient allocation can be decentralized as a competitive equilibrium and vice versa[22]. Therefore, we use terminologies interchangeably, e.g. prices in competitive equilibrium which corresponds to co-state variables in planner's problem.

The critical assumptions of our model include: (1) $b > \rho$, i.e. the rate of capital accumulation is larger than the discount rate, which makes capital accumulation profitable. (2) $a > 1$, i.e. capital deepening is costly. Capital will not be used in innovation unless it is necessary. (3) $\gamma > 1$, more advanced capital goods is labor saving, i.e. it requires less labor to produce one unit of final goods.

As proven below, under these three assumptions, the competitive equilibrium of the economy settles into a recurring cycle. It contains a *growth* phase, where two capitals of consecutive qualities are used and labor constantly reallocates from the less advanced technology to the more advanced one, and a *build-up* phase, where a new capital is accumulated before its price decreases to a level that make profitable of introducing it into production.
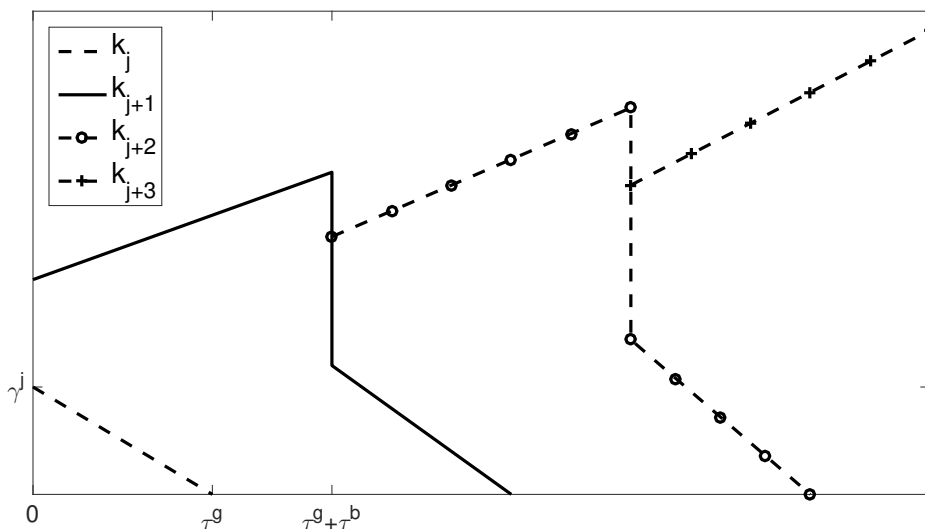
Figure 3.1: Evolution of capital stock

Figure 3.1 illustrates the evolution of capital stocks. Start with a point of time, normalized as $t = 0$, when capital of vintage j and vintage $j + 1$ are used in producing consumption

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[22]The competitive equilibrium of the economy can be described in the usual way as the combination of consumer optimization and profit maximization.

goods. Consumption/output increases over time as capital j is constantly converted to the more advanced capital $j+1$ and more and more labor reallocates to the latter. This growth phase ends at time $t = \tau^g$, when capital $j+1$ absorbs all labor. As proved later, at $t = \tau^g$, capital $j+2$ will not be immediately introduced into producing consumption goods, as its price is too expensive to make production profitable. Firms optimally wait and keep accumulating capital goods, which decreases the price of the latter. New technology, i.e. capital $j+2$, will only be adopted when its price reaches a level that is low enough for firms to make a profit. Denote τ^g the length of this build-up phase. At the end of the build-up phase, i.e. at $t = \tau^g + \tau^b$, firms innovate by converting capital $j+1$ to capital $j+2$ and employ the latter in the consumption sector. Here begins a new and recurring cycle. During the build-up phase, total output is fixed as there is no new technology introduced in production. The growing-then-stagnant evolution of consumption is illustrated in Figure 3.2.

Figure 3.2: Evolution of consumption
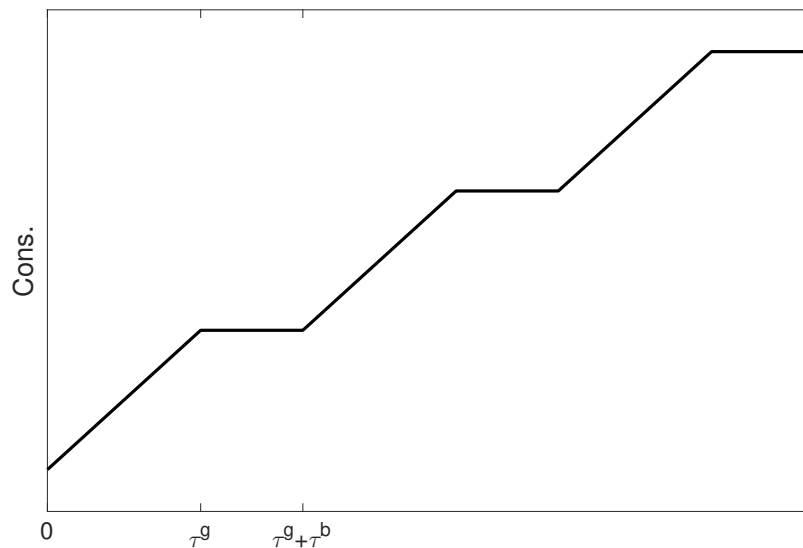
When capital j and $j+1$ are simultaneously used in producing consumption goods, the latter admits a higher capital income share, or equivalently lower labor income share, due to the labor-saving nature of technological progress. In the growth phase, as more labor shifts to capital $j+1$, the aggregate capital income share rises. Put differently, this process of labor reallocation increases labor productivity but not wage, therefore leading to a declining labor income share. In the build-up phase, the capital price and rental rate decline over time as capital self accumulates, and total consumption goods remains constant. As a result, the capital (labor) income share decreases (increases). Figure 3.3 illustrates the recurring and cyclical behavior of factor income shares.

Figure 3.3: Evolution of capital income share
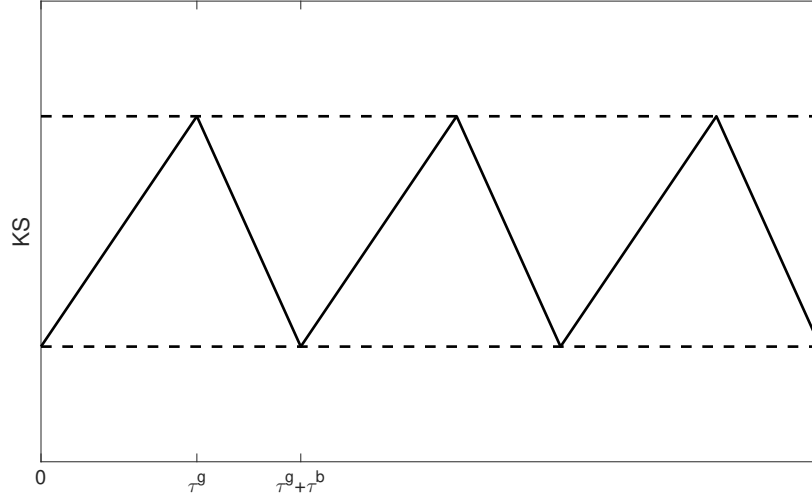
We now formally establish these properties of the competitive equilibrium outline above. Use marginal utility as the numeraire[23]. The price of period-t consumption goods is therefore, $1/c(t)$. Denote $q_j(t)$ the price of capital j in period t. Capital can be used for widening, that is, to create more capital of the same quality. The physical rate of return for capital widening is b. Zero profit for capital widening, when it occurs, implies that this return plus capital gains equal to the subjective discount rate, that is, $b + \dot{q}_{jt}/q_{jt} = \rho$, or equivalently[24],

$$\dot{q}_j(t)/q_j(t) = -(b - \rho) < 0.$$

The price of capital decreases over time as more is accumulated. The following proposition gives the level of capital prices.

Proposition 1: *No more than two qualities of knowledge capital are actually used to produce consumption, and these must be consecutive qualities. If j' is used to produce consumption, the price of capital j, $j > j'$*

$$q_j(t) \geq v_j(t) \equiv \frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}} \frac{1}{bc(t)}$$

with equality if j is also used to produce consumption.

Proof: see appendix.

The main part of the proof is to obtain the value/price of capital goods. Without loss of generality, assume capital j' and j, $j > j'$, are used in producing consumption goods. Zero

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[23]This way, the price of capital corresponds directly to the co-state variable associated with the law of motion for capital, in the current value Hamiltonian of planner's problem.

[24]This can also be derived from the Euler equation in planner's problem.

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profit conditions in the consumption sector imply

$$1 - r_j - \frac{w}{\gamma^j} = 0,$$

$$1 - r_{j'} - \frac{w}{\gamma^{j'}} = 0,$$

and the zero profit condition in the innovation sector leads to,

$$r_j = a^{j-j'} r_{j'}.$$

This is a system of three equations with three unknowns. It follows that

$$w = \gamma^{j'} \frac{a^{j-j'} - 1}{a^{j-j'} - 1/\gamma^{j-j'}},$$

$$r_j = \frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}},$$

and $r_{j'} = r_j / a^{j-j'}$. Rental rate is the flow value of capital stocks. This value, divided by b, i.e. the rate of capital accumulation, gives capital value/prices which are stock variables. We have, therefore, $v_j(t) = \frac{1}{c(t)} \frac{r_j(t)}{b} = \frac{1}{bc(t)} \frac{\gamma^{j'-j} - 1}{\gamma^{j-j'} - 1/a^{j-j'}}$. There is a $1/c(t)$ terms as it converts the unit of price from consumption goods to marginal utility.

One unit of capital can be used in either self accumulation at the rate of b, or production of consumption goods. If capital j is the only capital employed in producing consumption goods, before the labor supply constraint is reached, the price of capital j satisfies $q_j(t) = \frac{1}{bc(t)}$. This is larger than the price of capital j when capital j and j' are both used in production[25]. The reason is the following: 1 extra unit of capital j demands $1/\gamma^j$ units of labor, which requires replacement of $1/\gamma^{j'}$ units of capital j' currently used in production. This replacement effect makes capital j less valuable.

The replacement effect is also related to the result that there are at most two consecutive qualities of capital simultaneously used in production. When capital j' is employed in producing consumption goods, zero profit of innovation implies that the price of capital $j, j > j' + 1$ increases proportionately by $a^{j-j'}$. However, its value in production, due to the replacement effect, does not increase as much. It is therefore not profitable to adopt too advanced capital goods into production.

Proposition 2 summarizes the recurring cycles the economy settles into, and the cyclical behavior of factor shares.

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[25]The coefficient in the price formula of Proposition 1, $\frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}} < 1$.

Proposition 2: *Consumption grows at the rate $b - \rho$ during a growth phase which lasts for $\tau^g = \frac{\log \gamma}{b - \rho}$, followed by build-up phase during which consumption remaining constant, lasting $\tau^g = \frac{\log a}{b - \rho}$. The total length of a cycle is*

$$\tau^* = \frac{\log a + \log \gamma}{b - \rho}$$

The labor income share declines from $\frac{a-1}{a-1/\gamma}$ to $\frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$ in the growth phase, and increases back to $\frac{a-1}{a-1/\gamma}$ in the following build-up phase.

Proof: see appendix.

Consider a growth phase when capital j and $j+1$ are both used in production. From the price of k_{j+1}, and the fact that this price decreases at the rate $b - \rho$, consumption therefore grows at the rate $b - \rho$. At the end of the growth phase, k_{j+1} absorbs all labor force; and the price of k_{j+2} is a times that of k_{j+1} following the zero profit condition of innovation. As shown in appendix, this price is larger than the actual value of putting k_{j+2} in producing consumption goods. It is therefore not profitable to introduce k_{j+2} at the end of the growth phase. Instead, firms keep accumulating capital $j+1$[26], which decreases the capital price. This build-up phase ends when the (implicit) price of k_{j+2} equals its value in production. Then capital $j+2$ is introduced into the consumption sector, and a new growth phase begins.

In the growth phase, both k_j and k_{j+1} are used in sector 1, consumption grows at the rate $b - \rho$ as more and more labor reallocates from k_j to k_{j+1}. Capital $j+1$, by assumption more labor saving, admits a relatively lower labor share. The labor income share in firms employing k_j and k_{j+1} is

$$LS_j = \frac{w\ell_j}{\gamma^j \ell_j} = \frac{w}{\gamma^j} = \frac{a-1}{a-1/\gamma}$$

$$LS_{j+1} = \frac{w\ell_{j+1}}{\gamma^{j+1}\ell_{j+1}} = \frac{1}{\gamma}\frac{a-1}{a-1/\gamma}$$

LS_{j+1} is smaller than LS_j as $\gamma > 1$. The reallocation process in the growth phase thus decreases the aggregate labor income share. Equivalently, the capital income share in the growth phase increases from $\frac{1-1/\gamma}{a-1/\gamma}$ to $a\frac{1-1/\gamma}{a-1/\gamma}$.

In the build-up phase, consumption remains constant at γ^{j+1}. The rental price of capital, and therefore the capital income share in the consumption sector, decreases at the rate

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[26]Note that here we can alternatively assume that firms innovate and obtain capital $j+2$ immediately after the growth phase. What matters is that firms will not immediately employ capital $j+2$ in producing consumption goods. Actually we can assume firms do the innovation at any point of time during the build-up phase. All are equivalent in the sense that firms optimally choose the same time to start using capital $j+2$ in the consumption sector and the same amount of initial capital $j+2$ at that time.

$b - \rho$. The labor income share then increases at the rate $b - \rho$. As shown in the appendix, the factor income share at the end of a build-up phase is exactly the same as that in the beginning of a growth phase. It is worth to point out that the factor shares we focus here are that in the consumption sector. We do not explicit other two sectors as the capital income share is trivially 100% there. In the appendix, we calculated the factor income share in the whole economy and found a quite similar cyclical pattern.

Levels of capital and the initial phase   We have already proved that the economy eventually settles into a recurring cycle. However, we have not yet studied the levels of capital stock and the behavior at the very beginning of economy. Denote $j = 0$ the least advanced capital vintage, and τ_j the time when capital of vintage j is first employed in producing consumption goods. Without loss of generality, start with a growth phase when k_j and k_{j+1} are simultaneously used. Given an initial value of k_{j+1} at $t = \tau_{j+1}$, and law of motion for capital stock in the following growth and build-up phases, we can calculate k_{j+2} at $t = \tau_{j+2}$, i.e. the beginning of the next growth phase[27]. It turns out that $k_{j+2}(\tau_{j+2})$ and $k_{j+1}(\tau_{j+1})$ satisfy the following relation

$$\frac{k_{j+2}(\tau_{j+2})}{\gamma^{j+1}} = (a\gamma)^{\frac{\rho}{b-\rho}} \frac{k_{j+1}(\tau_{j+1})}{\gamma^j} - x.$$

where $x > 0$ is defined as $x \equiv a^{\frac{\rho}{b-\rho}} [\frac{\gamma}{\gamma-1} + \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}(\gamma^{\frac{\rho}{b-\rho}} - 1)]$. Figure 3.4 illustrates $\frac{k_{j+2}(t_{j+2})}{\gamma^{j+1}}$ as a function of $\frac{k_{j+1}(t_{j+1})}{\gamma^j}$. As $(a\gamma)^{\frac{\rho}{b-\rho}} > 1$, the function is steeper than a 45-*degree* line. There exists a unique steady state value for the normalized capital stock. An initial value of capital $j + 1$ below the steady state eventually leads to a negative capital stock; and any initial capital above the steady state level results in an explosion of capital stock.

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[27]We refer interested readers to the appendix for the details of calculation.

Figure 3.4: Steady state of normalized capital
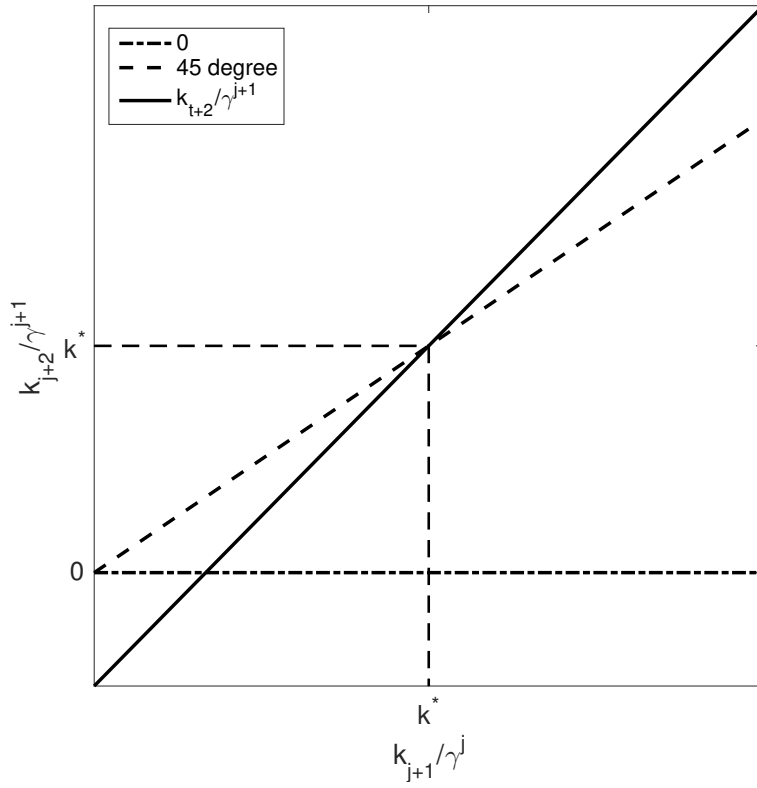
Focus on the case with $0 < k_0(0) < 1$, that is, the initial capital 0 is not high enough to employ all labor at $t = 0$. The first recurring cycle of the economy is when capital 0 and 1 are simultaneously used in sector 1. $k_1(\tau_1)$ should equal to the steady state value calculated above. At the initial unemployment phase, capital 1 is too expensive to be introduced immediately. The planner, or firms in the competitive equilibrium, optimally allocate $k_0^1(0)$ units of capital 0 in sector 1, and $k_0(0) - k_0^1(0)$ in sector 2. $k_0^1(t)$ and consequently consumption $c(t)$ grows over time until full employment at $t = \tau_0^g$, when $c(\tau_0^g) = 1$.

Then follows the initial build-up phase when capital 0 accumulates and consumption remains constant. This build-up phase ends at $t = \tau_1$ when the (implicit) price of capital 1 equals to its value in production, as in Proposition 1. At $t = \tau_1$, the economy enters the recurring cycles and behaves as described before. For the initial cycle, given an initial choice $k_0^1(0), 0 < k_0^1(0) < k_0(1)$, we can calculate the length of this cycle τ_1 and $k_0(t)$ for $0 < t \le \tau_1$. As shown in appendix, the equation $k_1(\tau_1) = k^*$ uniquely determines the initial choice, $k_0^1(0)$. Figure 3.5 illustrates the evolution of capital stock during the initial cycle.

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Figure 3.5: The initial cycle
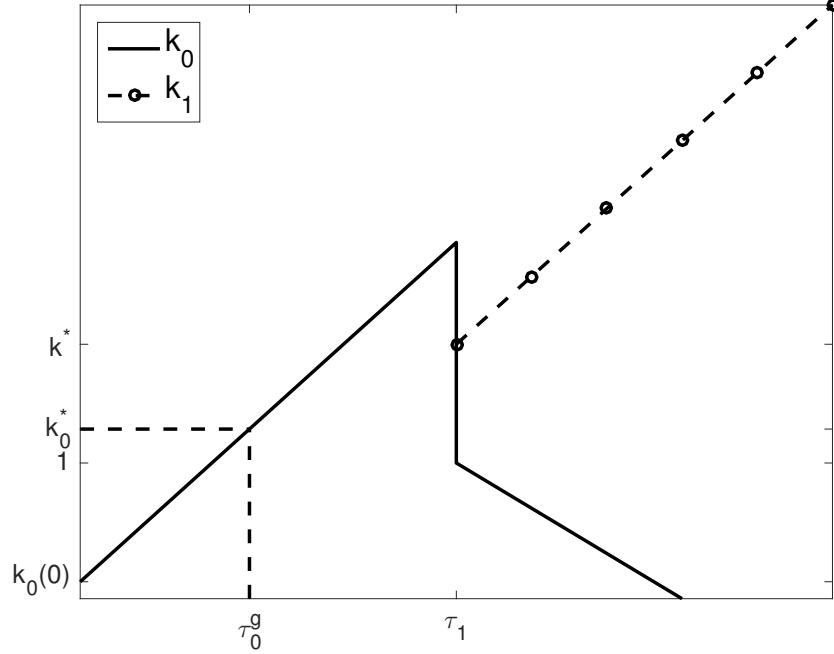
For $k_0(0) \geq 1$, the initial allocation choice is determined in the following way. Along the optimal path, convert the capital stock of different stocks into an equivalent units of k_0 by applying the rule that 1 unit of k_j is equivalent to $1/a^j$ units of k_0. The gives a continuous and increasing function of equivalent k_0 in time. For any initial value $k_0(0)$, we can find its corresponding point in this function, and determine the initial allocation accordingly.

Proposition 3 formally summarizes these results.

Proposition 3: *Depending on the initial value, there might be an initial phase when a single vintage of capital is employed and accumulated. After that initial phase, the economy settles into a recurring growth and build-up cycle. The value of capital stock j when it is first introduce at* $t = \tau_j$ *satisfies* $k_{j+1}(\tau_{j+1}) = \gamma^j k^*$, *where* k^* *is defined as*

$$k^* = \frac{x}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1},$$

with $x \equiv a^{\frac{\rho}{b-\rho}} [\frac{\gamma}{\gamma-1} + \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)} (\gamma^{\frac{\rho}{b-\rho}} - 1)]$.

3.1 Endogenous labor supply

In this subsection, we relax the assumption that labor supply is exogenous. An endogenous labor supply seems more appropriate over the business cycle frequency. As shown

below, new insights with endogenous labor supply include: in a growth phase, total employment decreases with a stagnant wage[28]; in the following build-up phase, households facing a rising wage optimally work more, as a result consumption also grows over time though at a lower rate than the growth phase. Further, the length of growth and build-up phase changes with endogenous labor supply while the total length of a cycle remains constant. The cyclical pattern of factor shares is as in the case of exogenous labor supply.

Formally, we endogenize labor supply by adding disutility of work into the utility function

$$\int_0^\infty e^{-\rho t}[\log c(t) - \zeta \frac{\eta - 1}{\eta} \ell(t)^{\frac{\eta}{\eta-1}}]dt,$$

where $\zeta > 0$ and $\eta > 1$. The first order condition w.r.t. working hours $\ell(t)$ is

$$\frac{w(t)}{c(t)} = \zeta \ell(t)^{\frac{1}{\eta-1}}$$

Note that wage is determined by zero profit in sector 1 and 2. In the growth phase, wage, in units of current consumption goods, is constant. The growth rate in consumption and working hours satisfy

$$-\frac{\dot{c}(t)}{c(t)} = \frac{1}{\eta - 1}\frac{\dot{\ell}(t)}{\ell(t)}.$$

As consumption grows at the rate $b - \rho$ during a growth phase, determined by zero profit of capital widening, working hours shrink at the rate $(\eta - 1)(b - \rho)$.

In the build-up phase, consumption will not remain constant with endogenous labor supply, as a rising wage encourages workers to supply more hours which increases production. Without loss of generality, focus on the build-up phase when only capital $j + 1$ is used in production. Substitute the production relation, $c(t) = \gamma^{j+1}\ell(t)$, into the first order condition for working hours, and we have

$$w(t) = \zeta \gamma^j \ell(t)^{\frac{\eta}{\eta-1}}.$$

Therefore, working hours during the build-up phase grow at the rate $(\eta - 1)/\eta$ times the growth rate of wage $w(t)$. As wage grows in the build-up phase, working hours and consequently consumption increase over time. Adding endogenous labor supply also changes the relative length of the growth and build-up phase, while the keeping the total length of cycle constant. Formally, we have the following proposition[29]

Proposition 4: *The economy with endogenous labor supply settles into a recurring cycle, consisting of a growth phase when consumption grows at the rate $b - \rho$ and a build-up phase when*

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[28]A combination of increasing output and non-increasing employment in the initial phase of a post-recession expansion resembles the job-less recovery phenomenon observed especially in recent recessions.

[29]Again, we refer interested readers to appendix for details of proof.

consumption grows at the rate $\frac{\frac{\eta-1}{\eta}\log\gamma}{\log a + \frac{\eta-1}{\eta}\log\gamma}(b-\rho)$. *A growth phase lasts for* $\tilde{\tau}^g = \frac{\log\gamma}{\eta(b-\rho)}$, *which is followed by a build-up phase lasting* $\tilde{\tau}^g = \frac{\log a + \frac{\eta-1}{\eta}\log\gamma}{b-\rho}$. *The total length of a cycle is*

$$\tilde{\tau}^* = \frac{\log a + \log \gamma}{b - \rho}.$$

Further, both the labor income share and labor supply decline in the growth phase, and increase during the build-up phase.

Proof: see appendix.

4 Discussion

In this section, we further discuss two related issues: interpretation of recessions in our model, and medium-term cycles.

Recessions and de-trending   Our mode features a phase with fast growth of output, and a following phase when consumption grows slow or stagnant, depending on if labor supply is endogenous or not. There is not a period when consumption drops as in recessions in real world. In our model, recessions are a point at the end of the build-up phase when firms innovate, scrape old technology and replace it with more advanced ones. Output does not fall in recessions as our model is deterministic, agents are perfect foresight, and there is no aggregate shocks in the economy.

Recessions are relative terms comparing to a trend. In our growth-then-stagnant economy, as shown in Figure 3.2, if we take away a trend component either through a H-P filter or others, there are booms and recessions relative to that trend. In particular, in quarters from the later part of a build-up phase to the earlier part of the following growth phase, there is negative growth in output relative to the trend, which correspond to recessions in data.

Medium-term cycles   While the factor income share displays strong variations over the business cycle frequency, as shown in Figures 6.2 and 6.6, the trend component also displays cycles at the medium-term frequency. For example, from the late 1950s to middle 1960s, there are a few moderate recessions and the capital income share fluctuates over business cycles, the trend term, however, is monotonically increasing. The same occurs from the early 1970s to middle 1980s.

Following Comin and Gertler (2006), we have extracted the medium-term cycles of factor income shares by applying a Baxter-King band pass filter, with a lower limit frequency of 2 quarters and an upper limit frequency of 200 quarters (50 years). Figure 6.5 actually shows that the BK trend component is quite close to the H-P trend with a smooth

parameter of 1600. We divide the whole period into two subsets as before, but now according to its trend component instead of the capital share series itself: quarters when the *trend component* of capital income share increases, and quarters when it decreases. Table 6.2 in appendix summarizes the behavior of labor productivity etc. over these two phases, which turns out to largely agree with that in Table 2.2. This agreement confirms that, while the focus of our paper is on the interaction of factor prices and technological progress over the business cycle, the mechanism we propose also applies to cycles in longer runs.

5 Conclusion

The factor shares have demonstrated a strong counter-cyclical pattern. A typical expansion begins with an increase in the capital income share; this share peaks substantially earlier than output, and falls in the the last phase of expansion. In periods of rising capital income shares, labor productivity and output grow at a fast rate while employment and wage increase much slower. The opposite is observed in periods when the capital income share declines.

In this paper, we provide a theory of why this may be due to the pattern of technological innovation under competition. Central to our theory are endogenous movements in relative factor prices creating incentives for replacing old technologies with new ones. At the beginning of an expansion, firms will pick new technologies that are labor-saving relatively to previous ones. Labor moves accordingly and its productivity increases faster than wages, hence the capital share and output increase rapidly. However, as the replacement process completes and more and more labor is employed, wages will eventually go up, drying the corporate profits, and finishing with it the expansion. This endogenous interaction between changes in the relative price of labor and labor-saving innovations generates simultaneously growth and cycles.

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6   Appendix

Proof of proposition 1   We have already calculated wage and interest rates when there are two vintages of capital employed in production. Here we show how to derive the price formula, and prove why there are at most two consecutive qualities of capital employed in producing consumption goods. Denote $p_j(t)$ the price of capital j in units of period-t consumption goods, and m_t the price of period t consumption goods in units of period 0 marginal utility, i.e. $m(t) = e^{-\rho t}\frac{1}{c(t)}$. The following non-arbitrage condition holds,

$$\frac{m(t+\Delta)}{m_t}p_j(t+\Delta) - p_j(t) = r_j(t), \quad \text{as } \Delta \to 0$$

It follows that

$$r_j(t) = \frac{m(t) + \Delta m(t)}{m(t)}[p_j(t) + \Delta p_j(t)] - p(t)$$

$$= \frac{\dot{m}(t)}{m(t)}p_j(t) + \dot{p}_t, \quad \text{as } \Delta \to 0$$

$$= p_j(t)[\frac{\dot{m}(t)}{m(t)} + \frac{\dot{p}_j(t)}{p_j(t)}]$$

$$= p_j(t)b$$

The last line follows as we know that capital price, in units of period-0 marginal utility, decreases at the rate of b. Therefore $p_j(t) = \frac{r_j(t)}{b}$. The price of capital j in units of period-t consumption goods, $q_j(t)$, is

$$q_j(t) = \frac{1}{c(t)}p_j(t) = \frac{1}{bc(t)}\frac{\gamma^{j'-j}-1}{\gamma^{j-j'}-1/a^{j-j'}}$$

Alternatively, we can calculate capital price using Hamiltonian. Recall that zero profit conditions in the consumption sector implies that

$$c(t) = r_j(t)k_j^1(t) + r_{j'}(t)k_{j'}^1(t) + w(t)$$

$$= r_j(t)[k_j^1(t) + \frac{1}{a^{j-j'}}k_{j'}^1(t)] + w(t)$$

The corresponding Hamiltonian is[30]

$$\mathcal{H} = \log c(t) + \lambda_j(t)b[k_j(t) - k_j^1(t)] + \lambda_{j'}(t)b[k_{j'}(t) - k_{j'}^1(t)]$$

$$= \log c(t) + \lambda_j(t)b[k_j(t) - k_j^1(t)] + \lambda_j(t)\frac{1}{a^{j-j'}}b[k_{j'}(t) - k_{j'}^1(t)]$$

$$= \log c(t) + \lambda_j(t)b[k_j(t) + k_{j'}(t)\frac{1}{a^{j-j'}} - k_j^1(t) - k_{j'}^1(t)\frac{1}{a^{j-j'}}]$$

$$= \log c(t) + \lambda_j(t)b[k_j(t) + k_{j'}(t)\frac{1}{a^{j-j'}} - \frac{1}{r_j(t)}c(t) - \frac{1}{r_j(t)}w(t)]$$

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[30]Note that any positive amount of capital j in sector 3, $k_{j'}^3$, appears as a negative term in the law of motion for $k_{j'}$ and a positive term in that for k_j. These two terms exactly cancel each other.

The first order condition *w.r.t.* c_t gives

$$\lambda_j(t) = \frac{1}{bc(t)} r_j(t) = \frac{1}{bc(t)} \frac{\gamma^{j-j'} - 1}{\gamma^{j-j'} - 1/a^{j-j'}}$$

To see why there are at most two vintages of capital used in production and they must be of consecutive quality, consider the case where capital j' is used in production. The price of capital $j' + 1$ is therefore $q_{j'+1}(t) = \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$. Zero profit condition of innovation dictates that for any capital $j, j > j' + 1$, its price equals $a^{j-j'-1} q_{j'+1}(t) = a^{j-j'-1} \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$, which is larger than its value in production, $\frac{\gamma^{j-j'}-1}{\gamma^{j-j'}-1/a^{j-j'}} \frac{1}{bc(t)}$. Therefore, any capital of vintage larger than $j' + 1$ will not be employed in production.

When capital j and $j + 1$ are simultaneously employed in production, as it is efficient to further accumulate capital $j + 1$ and replace capital j. Part of capital $j + 1$ will be used for self accumulation. Therefore $q_{j+1}(t)$ decreases at the rate of $b - \rho$, which implies that consumption grows at the rate of $b - \rho$.

Proof of proposition 2 Without loss of generality, consider a growth phase when capital j and $j + 1$ are both used in producing consumption goods. At the end of the growth phase, capital $j + 1$ absorbs all labor force, and the price of capital $j + 1$ according to proposition 2 is, $q_{j+1}(t) = \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$. At this point, zero profitability of innovation implies that the price of capital $j + 2$ satisfies $q_{j+2}(t) = aq_{j+1}(t) = a\frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$. However, the value of employing in producing consumption goods, by applying proposition 2 again, is $v_{j+2}(t) = \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(t)}$. As

$$q_{j+2}(t) = a\frac{\gamma - 1}{\gamma - 1/a} \frac{1}{bc(t)} \geq v_{j+2}(t) = \frac{\gamma - 1}{\gamma - 1/a} \frac{1}{bc(t)},$$

it is not profitable to introduce capital $j + 2$ into production at the end of the growth phase. Capital $j + 1$ will further accumulates, which decreases price of capital (of vintage $j + 1$ as well as $j + 2$). The left hand side in the above inequality decreases at the rate of $b - \rho$ while its right hand side remains constant. Capital $j + 2$ will be introduced into production when the LHS decreases and equals RHS.[31]. As the price of capital decreases at the rate of $b - \rho$, this build-up phase lasts for $\frac{\log a}{b-\rho}$. As for the growth phase, consumption grows from γ^j to γ^{j+1} at the rate of $b - \rho$. The growth phase therefore lasts for $\frac{\log \gamma}{b-\rho}$.

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[31]A different, and more technical, interpretation of this (in-)equality is, the original optimal control problems can be divided into a series of sub-problems, each dealing with the optimization problem for the length of period when two consecutive capital goods are used. Denote $\lambda_j(J)$ and $\lambda_{j+1}(J)$ the co-state variables for the dynamics of capital j and $j + 1$, respectively, in the sub-problem when capital j and $j + 1$ are simultaneously used in production. A necessary condition for equivalence of the original problem and the series of sub-problems is that $\lambda_{j+1}(J) = \lambda_{j+1}(J + 1)$, which is essentially the price condition here.

Levels of capital stock   We now investigate the evolution of capital stock. First calculate how much capital is transformed into that of a more advanced vintage at the beginning of a growth phase once the economy enters the recurring cycles. Consider a growth phase when capital j and $j+1$ are simultaneously employed in production. At the beginning of that phase, $a * k_j(t_0)$ units of capital j is converted into $k_{j+1}(t_0)$ units of capital $j+1$. Note that to guarantee the continuity of consumption, the remaining capital j used in producing consumption goods, $k_j^1(t_0) = \gamma^j$, and total consumption is $c(t_0) = \gamma^j$. Without loss of generality, normalize $t_0 = 0$[32].

During the growth phase, denote $\sigma_j(t)$ the fraction of labor employed by capital j,

$$\gamma^j \sigma_j(t) + \gamma^{j+1}(1 - \sigma_j(t)) = c(t)$$

It follows that $\sigma_j(t) = \frac{\gamma^{j+1} - c(t)}{\gamma^{j+1} - \gamma^j} = \frac{\gamma^{j+1} - \gamma^j e^{(b-\rho)t}}{\gamma^{j+1} - \gamma^j} = \frac{\gamma - e^{(b-\rho)t}}{\gamma - 1}$, where the second equality holds as consumption in the growth phase increases at the rate of $b - \rho$. Note that when $t = \frac{\log \gamma}{b-\rho}$, $\sigma_j(t) = 0$. That is, at the end of growth phase, all labor reallocates from capital j to $j+1$.

Assume that capital j is converted to capital $j+1$ as soon as it is freed from use in producing consumption goods during the growth phase[33]. That is,

$$k_j^3(t) = -dk_j(t) = -\gamma^j * d\sigma_j(t)$$
$$= \gamma^j \frac{b - \rho}{\gamma - 1} e^{(b-\rho)t} * dt$$

Therefore, during the growth phase, the law of motion for capital $j+1$ is[34]

$$dk_{j+1}(t) = bk_{j+1}^2(t)dt - k_{j+1}^3(t) + \frac{k_j^3(t)}{a}$$
$$= b[k_{j+1}(t) - \gamma^{j+1}(1 - \sigma_j(t))]dt - 0 + \frac{1}{a} * \gamma^j \frac{b - \rho}{\gamma - 1} e^{(b-\rho)t} dt$$

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[32]Without normalization, one can simply add all time variables in this subsection by the initial value, and all results here remain.

[33]Alternatively, we can assume that capital j released from production is first self-accumulated from time t for a period of positive length Δ_t, and converted to capital $j+1$ altogether at Δ_t. These two assumptions are equivalent in the sense that they deliver exactly the same amount of vintage $j+1$ capital goods at time $t + \Delta_t$.

[34]Note that here we assume that before capital $j+2$ is used in producing consumption goods, say at $t = \bar{t}$, capital $j+1$ will only be used in (producing consumption goods and) replicating itself, and not be used in creating capital $j+2$. Alternatively, we can assume that any capital $j+1$ beyond the necessary amount in producing consumption goods is converted immediately to capital $j+2$. The amount of capital $j+2$ obtained at $t = \bar{t}$ under two assumptions would be the same. In addition, as capital $j+2$ will not be used in producing consumption goods before $t = \bar{t}$, the price of capital $j+1$ is determined as before, and the (implied) price of capital $j+2$ is also not altered.

Equivalently,

$$\dot{k}_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}(1 - \sigma_j(t))] + \gamma^j \frac{b-\rho}{a(\gamma-1)} e^{(b-\rho)t}$$

$$= b[k_{j+1}(t) + \frac{\gamma^{j+1}}{\gamma-1}] + \gamma^j \frac{b(1-a\gamma)-\rho}{a(\gamma-1)} e^{(b-\rho)t}$$

until $t = \tau^g \equiv \frac{\log \gamma}{b-\rho}$ when the growth phase ends. The solution to this ordinary differential equation has the following form: $k_{j+1}(t) = \theta_0 + \theta_1 e^{bt} + \theta_2 e^{(b-\rho)t}$. Differentiating both sides w.r.t. time t and matching coefficients in common terms, we have

$$\theta_0 = -\frac{\gamma^{j+1}}{\gamma-1},$$

and

$$\theta_2 = \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)}.$$

Substituting these back into the formula for $k_{j+1}(t)$,

$$k_{j+1}(t) = -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 e^{bt} + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)} e^{(b-\rho)t}.$$

Using the initial condition at time $t = 0$,

$$k_{j+1}(0) = -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)},$$

we have,

$$\theta_1 = k_{j+1}(0) - \gamma^j \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}.$$

At time $t = \tau^g \equiv \frac{\log \gamma}{b-\rho}$,

$$k_{j+1}(\tau^g) = -\frac{\gamma^{j+1}}{\gamma-1} + \theta_1 \gamma^{\frac{b}{b-\rho}} + \gamma^j \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)} \gamma.$$

The build-up phase comes next and lasts until $t = \frac{\log \gamma + \log a}{b-\rho}$. During the build-up phase, $k_j^3(t) = 0$ as capital j has been used up; and $k_{j+1}^1 = \gamma^{j+1}$ as labor is all and only employed by capital $j+1$. Assume $k_{j+1}^3(t) = 0$.[35] The dynamics for $k_{j+1}(t)$ is

$$dk_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}]dt.$$

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[35]Note that here we made the assumption that, before $t = \frac{\log \gamma + \log a}{b-\rho}$, capital $j+1$ is only used in replicating itself and not used in creating $j+2$. Both activities satisfy zero profit conditions. In essence, between $t = \frac{\log \gamma}{b-\rho}$ and $t = \frac{\log \gamma + \log a}{b-\rho}$, various arrangements regarding what percentage of and when non-production capital $j+1$ is converted into capital $j+2$ are equivalent as they generate the same amount of capital $j+2$ at $t = \frac{\log \gamma + \log a}{b-\rho}$.

Solve this differential equation, and capital $j + 1$ satisfies

$$k_{j+1}(t) = \gamma^{j+1} + e^{b(t-\tau^g)}[k_{j+1}(\tau^g) - \gamma^{j+1}], \quad \text{for } \tau^g \leq t \leq \tau^g + \tau^b,$$

where $k_{j+1}(\tau^g)$ is the amount of capital $j + 1$ at $t = \tau^g$. When $t = \tau^g + \tau^b = \frac{\log \gamma + \log a}{b - \rho}$, capital $j + 1$ is

$$k_{j+1}(\tau^g + \tau^b) = \gamma^{j+1} + a^{\frac{b}{b-\rho}}[k_{j+1}(\tau^g) - \gamma^{j+1}].$$

At time $t = \tau^g + \tau^b$, γ^{j+1} units of capital are employed in producing consumption goods, and the remaining capital of vintage $j + 1$, $k_{j+1}(\tau^g + \tau^b) - \gamma^{j+1}$, is converted to capital of vintage $j + 2$. It follows that,

$$\begin{aligned} k_{j+2}(\tau^g + \tau^b) &= \frac{1}{a}[k_{j+1}(\tau^g + \tau^b) - \gamma^{j+1}] \\ &= \frac{a^{\frac{b}{b-\rho}}}{a}[k_{j+1}(\tau^g) - \gamma^{j+1}] \\ &= a^{\frac{\rho}{b-\rho}}\gamma - \frac{\gamma^{j+1}}{\gamma - 1} + [k_{j+1}(0) - \gamma^j\tilde{x}]\gamma^{\frac{\rho}{b-\rho}} + \gamma^j\tilde{x} \end{aligned}$$

where $\tilde{x} \equiv \frac{(\alpha\gamma - 1)b + \rho}{\rho\alpha(\gamma - 1)}$. Equivalently,

$$\begin{aligned} \frac{k_{j+2}(\tau^g + \tau_b)}{\gamma^{j+1}} &= a^{\frac{\rho}{b-\rho}}\left\{-\frac{\gamma}{\gamma - 1} + [\frac{k_{j+1}(0)}{\gamma^j} - x]\gamma^{\frac{\rho}{b-\rho}} + x\right\}, \\ &= (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - a^{\frac{\rho}{b-\rho}}[\frac{\gamma}{\gamma - 1} + \tilde{x}(\gamma^{\frac{\rho}{b-\rho}} - 1)]. \end{aligned}$$

This is the formula we obtain in text. From this equation, there is a unique steady state value of the normalized capital stock, $k* \equiv \frac{k_{j+1}(\tau_{j+1})}{\gamma^j}$, with τ_{j+1} the first time capital $j + 1$ used in production, which satisfies[36]

$$k^* = \frac{(a\gamma - 1)(b - \rho)}{\rho a(\gamma - 1)} \frac{(a\gamma)^{\frac{\rho}{b-\rho}} - a^{\frac{\rho}{b-\rho}}}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1}.$$

Denote $j = 0$ the least advanced capital. The economy enters a recurring cycle when capital of vintage 1 is created and employed in production. Denote τ_1 the first time capital 1

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[36]Note that

$$k_{j+1}(\tau^g) = \gamma^{j+1}\left[1 + \frac{(a\gamma - 1)(b - \rho)}{\rho a(\gamma - 1)} \frac{(\gamma^{\frac{\rho}{b-\rho}} - 1)}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1}\right].$$

$k_{j+1}(\tau^g) > k_{j+1}(0)$ requires

$$\gamma > \frac{(a\gamma - 1)(b - \rho)}{\rho a(\gamma - 1)} \frac{\gamma^{\frac{\rho}{b-\rho}} - 1}{(a\gamma)^{\frac{\rho}{b-\rho}} - 1}(a^{\frac{\rho}{b-\rho}} - \gamma).$$

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is used in production, $k_1(\tau_1) = k^*$. The value of $k_1(\tau_1)$ is determined in the initial growth and build-up cycle, which we turn to next.

The initial cycle Denote $k_0(0)$ the initial value of capital of vintage 0. Start with the case $0 < k_0(0) < 1$. That is, there is not enough initial capital to employ all labor force. We need to determine how to allocate initial capital between producing consumption goods and self-accumulation at $t = 0$. Denote $k_0^1(0)$ units of capital allocated in producing consumption goods.

Note that during this phase, the price of capital in terms of current marginal consumption is $q_0(t) = \frac{1}{bc(t)}$[37]. The rental price of capital is 1, and wage is 0, both in units of current consumption goods. The production function is $c(t) = \min\{k_0^1(t), \ell_0(t)\}$. During this stage, consumption grows at the rate of $b - \rho$. The dynamics of $k_0(t)$ is

$$\dot{k}_0(t) = b[k_0(t) - k_0^1(t)]$$
$$= b[k_0(t) - k_0^1(0)e^{(b-\rho)t}]$$

The solution to this ODE is of the form: $k_0(t) = \phi_0 + \phi_1 e^{bt} + \phi_2 e^{(b-\rho)t}$. Differentiating this ODE and matching coefficients with the formula above gives

$$\phi_0 = 0, \quad \phi_2 = k_0^1(0)\frac{\rho}{b}$$

Further use the initial condition to obtain $\phi_1 = k_0(0) - \frac{b}{\rho}k_0^1(0)$. This initial growth phase stops at $c(\tau_0^g) = k_0^1(0)e^{(b-\rho)\tau_0^g} = 1$, that is, at $\tau_0^g = \frac{1}{b-\rho}\log\frac{1}{k_0^1(0)}$. The capital stock at $t = \tau_0^g$ is

$$k_0(\tau_0^g) = k_0(0) * k_0^1(0)^{\frac{-b}{b-\rho}} - \frac{b}{\rho}k_0^1(0)^{\frac{-\rho}{b-\rho}} + \frac{b}{\rho}.$$

The economy then enters the initial build-up phase where the dynamics of capital is given by

$$\dot{k}(t) = b[k(t) - 1].$$

The solution to this ODE is

$$k_0(t) = 1 + e^{b(t-\tau_0^g)}[k_0(\tau_0^g) - 1]$$

To determine the length of the initial build-up phase, note that the price of capital 0 at $t = \tau_0^g$ is $q_0(\tau_0^g) = \frac{1}{bc(\tau_0^g)} = \frac{1}{b}$. From the zero profit condition of innovation, the (implicit) price of capital 1 is $q_1^*(\tau_0^g) = aq_0(\tau_0^g) = \frac{a}{b}$. Denote τ_1 the first time when capital 1 is created and employed in production. The length of this build-up phase is therefore $\tau_1 - \tau_0^g$. The price of capital 1 at $t = \tau_1$ is $q_1(\tau_1) = \frac{\gamma-1}{\gamma-1/a}\frac{1}{b}$. As the capital price decreases at the rate of $b - \rho$ during the build-up phase, we have

$$q_1^*(\tau_0^g)e^{-(b-\rho)(\tau_1-\tau_0^g)} = q_1(\tau_1)$$

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[37]This is obtained from the Euler equation.

The length of this build-up phase is

$$\tau_1 - \tau_0^g = \frac{1}{b - \rho} \log(\frac{a\gamma - 1}{\gamma - 1})$$

Note that this is different from the length of a build-up phase after the economy enters the recurring cycles. At $t = \tau_1$, the value of capital 0 is

$$k_0(\tau_1) = 1 + (\frac{a\gamma - 1}{\gamma - 1})^{\frac{b}{b - \rho}}[k_0(\tau_0^g) - 1],$$

Of which 1 unit is used in producing consumption goods, and the remaining $k_0(t_1) - 1$ units converted to capital of vintage 1. Therefore the amount of vintage-1 capital at $t = \tau_1$ is

$$\begin{aligned}
k_1(\tau_1) &= \frac{1}{a}(\frac{a\gamma - 1}{\gamma - 1})^{\frac{b}{b - \rho}}[k_0(\tau_0^g) - 1] \\
&= \frac{1}{a}(\frac{a\gamma - 1}{\gamma - 1})^{\frac{b}{b - \rho}}[k_0(0) * k_0^1(0)^{\frac{-b}{b - \rho}} - \frac{b}{\rho}k_0^1(0)^{\frac{-\rho}{b - \rho}} + \frac{b}{\rho} - 1] \\
&= \frac{1}{a}(\frac{a\gamma - 1}{\gamma - 1})^{\frac{b}{b - \rho}}\left\{k_0(0)^{\frac{-\rho}{b - \rho}}\chi^{\frac{-\rho}{b - \rho}}[1 - \frac{b}{\rho}x] + \frac{b}{\rho} - 1\right\}
\end{aligned}$$

where $\chi \equiv \frac{k_0^1(0)}{k_0(0)}$ is the fraction of initial capital that is used in producing consumption goods. The steady state condition we derived before requires $k_1(\tau_1) = k^*$. Note that $k_1(\tau_1)$ is a strictly decreasing function of $\chi \equiv \frac{k_0^1(0)}{k_0(0)}$. As $x \to 0$, $k_1(t_1) \to \infty$. On the other hand, as $x \to 1$, $k_1(t_1) \to \frac{1}{a}(\frac{a\gamma-1}{\gamma-1})^{\frac{b}{b-\rho}}(\frac{b}{\rho} - 1)(1 - k_0(0)^{\frac{-\rho}{b-\rho}}) < 0$. The monotonicity of $k_1(\tau_1)$ guarantees existence and uniqueness of a $k_0^1(0)$ that satisfies the steady state condition.

Factor shares in the whole economy   Note that so far we focus on factor income shares in the consumption sector, instead of the whole economy. This is justified by the fact that the investment sector has a zero labor income share, or equivalently 100% capital income share. Adjusting the factor income share accordingly change its *levels*, but does not affect *trend*. To see this point, consider a growth phase where capital j and $j + 1$ are simultaneously used in production. Denote t_{j+1} the first time capital $j + 1$ is created and employed in production. At $t = t_{j+1}$, the gross labor income share in the whole economy is

$$\begin{aligned}
\tilde{LS}(t_{j+1}) &= \frac{w(t_{j+1})}{c(t_{j+1}) + q_{j+1}(t_{j+1})c(t_{j+1})k_{j+1}(t_{j+1})}, \\
&= \frac{w(t_{j+1})/c(t_{j+1})}{1 + \frac{\gamma - 1}{\gamma - 1/a}\frac{\gamma^j}{bc(t_{j+1})}\frac{k_{j+1}(t_{j+1})}{\gamma^j}}.
\end{aligned}$$

$\frac{w(t_{j+1})}{c(t_{j+1})}$ is the labor share in the consumption goods producing sector. As $\frac{\gamma^j}{c(t_{j+1})} = 1, \forall j$, and $\frac{k_{j+1}(t_{j+1})}{\gamma^j}$ is a constant in steady state, The aggregate labor share at $t = t_{j+1}$ is independent of capital vintages.

For $t > t_{j+1}$,

$$\tilde{LS}(t) = \frac{w(t)}{c(t) + q_{j+1}(t)c(t)\dot{k}_{j+1}(t)}$$

where $q_{j+1}(t)$ is price of vintage $j+1$ capital/investment goods in terms of time-t marginal utility, and $1/c(t)$ is the price of time-t consumption in terms of time-t marginal utility; and $q_{j+1}(t)c(t)$ is the price of capital $j+1$ in terms of time-t consumption goods. $\dot{k}_{j+1}(t)$ is the gross investment at time t. Note that

$$\begin{aligned}
\tilde{LS}(t) &= \frac{w(t)}{c(t) + q_{j+1}(t)c(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + q_{j+1}(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^j}{bc(t)}[b(\frac{k_{j+1}(t_{j+1})}{\gamma^j} - x)e^{b(t-t_{j+1})} + (b-\rho)xe^{(b-\rho)(t-t_{j+1})}]}
\end{aligned}$$

where $x \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$ is a constant. Note $c(t) = \gamma^j e^{(b-\rho)(t-t_{j+1})}$. $\frac{k_{j+1}(t_{j+1})}{\gamma^j}$ is a constant in steady state. Therefore the denominator is a function of $t - t_{j+1}$, and independent of j itself. We have already shown that the numerator, $w(t)/c(t)$, which is the labor share in the consumption production sector, does not depend on j. Therefore, the aggregate labor share is also independent of capital vintages.

During the build-up phase, the price of vintage $j+1$ capital/investment goods, in terms of consumption goods, decreases at the rate of $b - \rho$. Consumption remain stagnant at γ^{j+1}. The aggregate labor share is

$$\begin{aligned}
\tilde{LS}(t) &= \frac{w(t)}{c(t) + q_{j+1}(t)c(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + q_{j+1}(t)\dot{k}_{j+1}(t)} \\
&= \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^j}{bc(t)}e^{-(b-\rho)(t-t_{j+1}^g)}[be^{b(t-t_{j+1}^g)}(\frac{k_{j+1}(t_{j+1}^g)}{\gamma^j} - \gamma)]} \\
&= \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^{j+1}}{bc(t)}[be^{\rho(t-t_{j+1}^g)}(-\frac{\gamma}{\gamma-1} + (\frac{k_{j+1}(0)}{\gamma^j} - x)\gamma^{\frac{\rho}{b-\rho}} + x)]}
\end{aligned}$$

where $x \equiv \frac{(a\gamma-1)(b-\rho)}{\rho a(\gamma-1)}$ as above, and t_{j+1}^g denotes the end (beginning) time of the growth (build-up) phase. As in the growth stage, both numerator and denominator are indepen-

dent of capital vintages.

The previous calculation does not subtract capital depreciation from GDP. We now calculate the net labor income share, which is the ratio of total wage income to non-depreciation value added. Again, focus on the growth phase when capital j and $j + 1$ are employed in production and the following build-up phase; denote t_{j+1} the first time capital $j + 1$ is created and used in producing consumption goods. At $t = t_{j+1}$, the net labor income share is

$$\tilde{LS}^{net}(t_{j+1}) = \frac{w(t_{j+1})}{c(t_{j+1}) + q_{j+1}(t_{j+1})c(t_{j+1})k_{j+1}(t_{j+1}) - q_j(t_{j+1})c(t_{j+1})[k_j(t_{j+1}) - \gamma^j]}$$

where $k_j(t_{j+1}) - \gamma^j$ units of capital j is used to innovate on capital $j + 1$ and fully depreciated. The zero profit condition of innovation implies that $q_{j+1}(t_{j+1})k_{j+1}(t_{j+1}) - q_j(t_{j+1})[k_j(t_{j+1}) - \gamma^j] = 0$. Therefore, at $t = t_{j+1}$, the net labor income share equals to that in the consumption goods producing sector.

For $t > t_{j+1}$ and during the growth phase,

$$\tilde{LS}^{net}(t) = \frac{w(t)}{c(t) + q_{j+1}(t)c(t)\dot{k}_{j+1}(t) - q_j(t)c(t)\dot{k}_j(t)}$$

$$= \frac{w(t)/c(t)}{1 + q_{j+1}(t)\dot{k}_{j+1}(t) - \frac{q_{j+1}(t)}{a}\dot{k}_j(t)}$$

where $\dot{k}_{j+1}(t)$ is investment (in capital $j + 1$) and $\dot{k}_j(t)$ is depreciation (of capital j) at time t. The second equation follows from the zero profit condition, $\frac{q_j(t)}{a} = q_{j+1}(t)$. The dynamics of $k_{j+1}(t)$ is the same as before. For $\dot{k}_j(t)$, as calculated before,

$$\dot{k}_j(t) = \gamma^j \frac{b - \rho}{\gamma - 1} e^{(b-\rho)(t-t_{j+1})}$$

Substituting back into the net labor income share formula, we have

$$\tilde{LS}^{net}(t) = \frac{w(t)/c(t)}{1 + \frac{\gamma-1}{\gamma-1/a}\frac{\gamma^j}{bc(t)}[b(\frac{k_{j+1}(t_{j+1})}{\gamma^j} - x)e^{b(t-t_{j+1})} + (x - \frac{b-\rho}{a(\gamma-1)})e^{(b-\rho)(t-t_{j+1})}]}$$

where $x \equiv \frac{(a\gamma-1)b+\rho}{\rho a(\gamma-1)}$. As $c(t) = \gamma^j e^{(b-\rho)t}$, and $\frac{k_{j+1}(t_{j+1})}{\gamma^j}$ is a constant in steady state. The formula above is again independent of capital vintages.

In the following build-up phase, there is no capital depreciation as no innovation occurs in that phase. Therefore, the net labor income share in the whole economy is the same as the gross one calculated above.

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Figure 6.1 presents an illustrating example for 'the labor income share in the consumption sector', 'the gross labor income share in GDP', and 'the net labor income share in GDP'.

Figure 6.1: Labor share in the whole economy
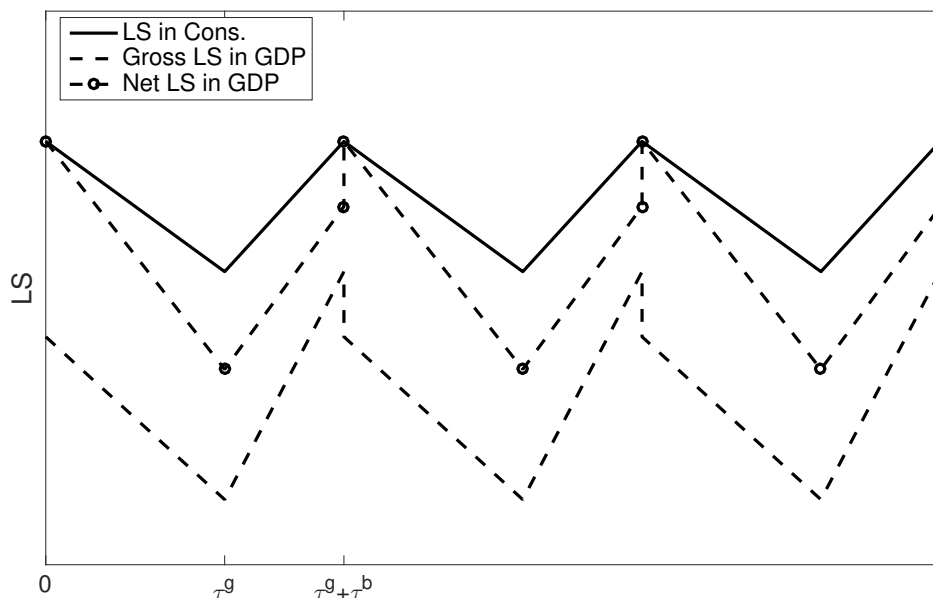
Endogenous labor supply Consider a growth phase in the recurring cycles when capital j and $j+1$ are simultaneously employed in production. At the beginning of that phase, normalized as $t = 0$, the following three conditions hold

$$c(0) = \gamma^j \ell(0);$$

$$\frac{w(0)}{c(0)} = \zeta \ell(0)^{\frac{1}{\eta-1}};$$

$$w(0) = \gamma^j \frac{a-1}{a-1/\gamma}.$$

Therefore, we have[38]

$$\ell(0) = [\frac{a-1}{a-1/\gamma} \frac{1}{\zeta}]^{\frac{\eta-1}{\eta}}, \qquad c(0) = \gamma^j \ell(0).$$

At $t = 0$, $a * k_j(0)$ units of capital j is converted into $k_{j+1}(0)$ units of capital $j+1$. The remaining capital j that is used in producing consumption goods, $k_j^1(t_0) = \gamma^j \ell(0)$, and total consumption is $c(0) = \gamma^j \ell(0)$.

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[38]assuming that $\frac{a-1}{a-1/\gamma} \frac{1}{\zeta} < 1$.

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During the growth phase, denote $\sigma_j(t)$ the fraction of labor employed by capital j, total production is

$$\gamma^j \ell(t) \sigma_j(t) + \gamma^{j+1} \ell(t)(1 - \sigma_j(t)) = c(t)$$

It follows that $\sigma_j(t) = \frac{\gamma^{j+1} - c(t)/\ell(t)}{\gamma^{j+1} - \gamma^j} = \frac{\gamma^{j+1} - \frac{c(0)}{\ell(0)} e^{\eta(b-\rho)t}}{\gamma^{j+1} - \gamma^j} = \frac{\gamma - e^{\eta(b-\rho)t}}{\gamma - 1}$, where the second equality holds as consumption in the growth phase increases at the rate of $b - \rho$, and hours decrease at the rate of $(\eta - 1)(b - \rho)$. Note that when $t = \frac{\log \gamma}{\eta(b-\rho)}$, $\sigma_j(t) = 0$. That is, with endogenous labor supply, the length of the growth phase shrinks from $\frac{\log \gamma}{b-\rho}$ to $\frac{\log \gamma}{\eta(b-\rho)}$.

At $t = \tau^g$, the labor supply, $\ell(\tau^g)$, is

$$\ell(\tau^g) = \ell(0) * e^{-(\eta-1)(b-\rho)\frac{\log \gamma}{\eta(b-\rho)}} = \ell(0) \gamma^{-\frac{\eta-1}{\eta}},$$

and the price of capital $j + 1$ in units of current marginal utility, $q_{j+1}(\tau_g)$, is[39],

$$q_{j+1}(\tau^g) = \frac{\gamma - 1}{\gamma - 1/a} \frac{1}{bc(\tau^g)},$$

Then comes the build-up phase, during which $q_{j+1}(t)$ still declines at the rate of $b - \rho$. However, with endogenous labor supply, $c(t)$ now increases over time in the build-up phase. The build-up phase ends at $t = \tau^g + \tau^b$ when the capital price satisfies

$$q_{j+1}(\tau^g + \tau^b) = \frac{1}{a} \frac{\gamma - 1}{\gamma - 1/a} \frac{1}{bc(\tau^g + \tau^b)}.$$

The length of the build-up phase satisfies

$$e^{(b-\rho)\tau^b} = a \frac{c(\tau^g + \tau^b)}{c(\tau^g)} = a \frac{\ell(\tau^g + \tau^b)}{\ell(\tau^g)}.$$

On the other hand, wage, in units of current consumption goods, grows from $w(\tau^g) = \gamma^j \frac{a-1}{a-1/\gamma}$ at the beginning of a build-up phase, to $w(\tau^g + \tau^b) = \gamma^{j+1} \frac{a-1}{a-1/\gamma}$. Combining with the condition $w(t) = \zeta \gamma^{j+1} \ell(t)^{\frac{\eta}{\eta-1}}$, for $t = \tau^g$ and $t = \tau^g + \tau^b$, we have

$$\frac{\ell(\tau^b + \tau^b)}{\ell(\tau^g)} = \gamma^{\frac{\eta-1}{\eta}}.$$

Notice the recurring nature of the problem, as in $\ell(\tau^g + \tau^b) = \ell(0)$. It follows that

$$\tau^b = \frac{\log a + \frac{\eta-1}{\eta} \log \gamma}{b - \rho},$$

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[39]The (implied) price of capital $j + 2$, $q_{j+2}(\tau^g)$, is, $q_{j+2}(\tau^g) = a * q_{j+1}(\tau^g) = a \frac{\gamma-1}{\gamma-1/a} \frac{1}{bc(\tau^g)}$.

which is longer than the case with exogenous labor supply. Recall that $\tau^g = \frac{\log \gamma}{\eta(b-\rho)}$, which is shorter with endogenous labor supply. The length of a whole cycle remains unchanged,

$$\tau^g + \tau^b = \frac{\log \gamma}{\eta(b-\rho)} + \frac{\log a + \frac{\eta-1}{\eta}\log \gamma}{b-\rho} = \frac{\log a + \log \gamma}{b-\rho},$$

The growth rate of consumption, as well as working hours, in the build-up phase satisfies

$$g^b = \frac{\frac{\eta-1}{\eta}\log \gamma}{\log a + \frac{\eta-1}{\eta}\log \gamma}(b-\rho).$$

This growth rate is smaller than the growth rate of consumption in the growth phase, which is $g^g = b - \rho$. On the other hand, the labor income share behaves the same as in the exogenous labor supply case. That is, it decreases in the growth phase, and increases in the build-up phase.

Then determine the capital stock. Assume that, during the growth phase, capital j is converted to capital $j+1$ as soon as it is freed from use in producing consumption goods. That is,

$$k_j^3(t) = -dk_j(t) = -\gamma^j * d[\ell(t)\sigma_j(t)]$$

$$= \gamma^j \ell(0)\frac{b-\rho}{\gamma-1}[\gamma(\eta-1)e^{-(\eta-1)(b-\rho)t} + e^{(b-\rho)t}] * dt$$

The law of motion for capital $j+1$ in the growth phase is

$$dk_{j+1}(t) = bk_{j+1}^2(t)dt - k_{j+1}^3(t) + \frac{k_j^3(t)}{a}$$

$$= b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))]dt - 0$$

$$+ \frac{1}{a} * \gamma^j \ell(0)\frac{b-\rho}{\gamma-1}[\gamma(\eta-1)e^{-(\eta-1)(b-\rho)t} + e^{(b-\rho)t}] * dt$$

Equivalently,

$$\dot{k}_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}(1-\sigma_j(t))] + \gamma^j \ell(0)\frac{b-\rho}{a(\gamma-1)}[\gamma(\eta-1)e^{-(\eta-1)(b-\rho)t} + e^{(b-\rho)t}]$$

$$= bk_{j+1}(t) + \frac{\gamma^j \ell(0)}{a(\gamma-1)}\left\{[b(1-a\gamma) - \rho]e^{(b-\rho)t} + \gamma[b(a+\eta-1) - \rho(\eta-1)]e^{-(\eta-1)(b-\rho)t}\right\}$$

The solution to this ODE is

$$k_{j+1}(t) = \theta_1 e^{bt} + \theta_2 e^{(b-\rho)t} + \theta_3 e^{-(\eta-1)(b-\rho)t},$$

with

$$\theta_1 = k_{j+1}(0) - \theta_2 - \theta_3;$$

$$\theta_2 = \frac{\gamma^j \ell(0)}{a\rho(\gamma-1)}[b(a\gamma-1) + \rho];$$

$$\theta_3 = \frac{\gamma^j \ell(0)}{a(\gamma-1)}\gamma\frac{ab + (b-\rho)(\eta-1)}{b-(\eta-1)(b-\rho)}.$$

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At time $t = \tau^g \equiv \frac{\log \gamma}{\eta(b-\rho)}$,

$$k_{j+1}(\tau^g) = \theta_1 \gamma^{\frac{b}{\eta(b-\rho)}} + \theta_2 \gamma^{\frac{1}{\eta}} + \theta_3 \gamma^{-\frac{\eta-1}{\eta}}$$

During the following build-up phase, the dynamics for $k_{j+1}(t)$ is

$$dk_{j+1}(t) = b[k_{j+1}(t) - \gamma^{j+1}\ell(t)]dt,$$
$$= b[k_{j+1}(t) - \gamma^{j+1}\ell(\tau^g)e^{g^b t}]dt.$$

The solution to this differential equation is

$$k_{j+1}(t) = [k_{j+1}(\tau^g) - \theta]e^{b(t-\tau^g)} + \theta e^{g^b(t-\tau^g)}, \quad \text{for } \tau^g \leq t \leq \tau^g + \tau^b,$$

with

$$\theta \equiv \frac{b}{b-g}\gamma^{j+1}\ell(\tau^g).$$

At $t = \tau^g + \tau^b = \frac{\log \gamma + \log a}{b-\rho}$, capital $j+1$ is

$$k_{j+1}(\tau^g + \tau^b) = k_{j+1}(\tau^g)[a\gamma^{\frac{\eta-1}{\eta}}]^{\frac{b}{b-\rho}} - \theta a\gamma^{\frac{\eta-1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}],$$

with $k_{t+1}(\tau^g)$ is determined in the growth phase. At time $t = \tau^g + \tau^b$, $\gamma^{j+1}\ell(0)$ units of capital is employed in producing consumption goods, and the remaining capital of vintage $j+1$, $k_{j+1}(\tau^g + \tau^b) - \gamma^{j+1}\ell(0)$, is converted to capital of vintage $j+2$. It follows that,

$$
\begin{aligned}
k_{j+2}(\tau^g + \tau^b) &= \frac{1}{a}[k_{j+1}(\tau^g + \tau^b) - \gamma^{j+1}\ell(0)] \\
&= \frac{1}{a}\left\{[(k_{j+1}(0) - \theta_2 - \theta_3)\gamma^{\frac{b}{\eta(b-\rho)}} + \theta_2\gamma^{\frac{1}{\eta}} + \theta_3\gamma^{-\frac{\eta-1}{\eta}}](a\gamma^{\frac{\eta}{\eta-1}})^{\frac{b}{b-\rho}} \right. \\
&\quad \left. - \theta a\gamma^{\frac{\eta-1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}] - \gamma^{j+1}\ell(0)\right\} \\
&= k_{j+1}(0)a^{\frac{\rho}{b-\rho}}\gamma^{\frac{b}{b-\rho}} - a^{\frac{\rho}{b-\rho}}\gamma^{\frac{b}{b-\rho}}[\theta_2(1 - \gamma^{-\frac{\rho/\eta}{b-\rho}}) + \theta_3(1 - \gamma^{-\frac{\eta b - (\eta-1)\rho}{\eta(b-\rho)}})] \\
&\quad - \theta a\gamma^{\frac{\eta-1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}] - \gamma^{j+1}\ell(0)
\end{aligned}
$$

Therefore,

$$\frac{k_{j+2}(\tau^g + \tau^b)}{\gamma^{j+1}} = (a\gamma)^{\frac{\rho}{b-\rho}}\frac{k_{j+1}(0)}{\gamma^j} - \bar{x},$$

with $\bar{x} > 0$ defined as

$$\bar{x} \equiv (a\gamma)^{\frac{\rho}{b-\rho}}[\frac{\theta_2}{\gamma^j}(1 - \gamma^{-\frac{\rho/\eta}{b-\rho}}) + \frac{\theta_3}{\gamma^j}(1 - \gamma^{-\frac{\eta b - (\eta-1)\rho}{\eta(b-\rho)}})] + \frac{\theta}{\gamma^j}a\gamma^{-\frac{1}{\eta}}[(a\gamma^{\frac{\eta-1}{\eta}})^{\frac{\rho}{b-\rho}} - \frac{1}{a}] + \gamma\ell(0).$$

Note that as there is a γ^j term in all θ, θ_2 and θ_3, x is independent of j.

The relation between $\frac{k_{j+2}(t_{j+2})}{\gamma^{j+1}}$ and $\frac{k_{j+1}(t_{j+1})}{\gamma^j}$ is essentially the same as in the exogenous labor supply case, as depicted in Figure 3.4. As $(a\gamma)^{\frac{\rho}{b-\rho}} > 1$, there exists a unique steady state value of capital. Further, any initial capital below the steady state (consuming too much at the beginning) will eventually leads to a negative capital stock; and any initial capital above the steady state level (consuming too little at the beginning) leads to an explosion of the capital stock.

We now move to the initial growth phase. Denote $k_0(0)$ the initial value of capital, and start with the case $0 < k_0(0) < \ell(0)$. We need to determine how much initial capital is used in production and how much in accumulating. Denote $k_0^1(0)$ the units of capital in producing consumption goods. The production function is $c(t) = \min\{k_0^1(t), \ell_0(t)\}$. The following relations hold,

$$c = k_0^1 = \ell_0; \quad 1 = r + w; \quad w = \zeta\ell_0^{\frac{\eta}{\eta-1}}.$$

Therefore, with a given $k_0^1(0)$, the rental prices are $w(0) = \zeta k_0^1(0)^{\frac{\eta}{\eta-1}}$, and $r(0) = 1 - w(0)$. The implied price of capital 0 is

$$q_0(0) = [1 - \zeta k_0^1(0)^{\frac{\eta}{\eta-1}}]\frac{1}{bc(0)},$$

and

$$q_0(t) = [1 - \zeta k_0^1(t)^{\frac{\eta}{\eta-1}}]\frac{1}{bc(t)}, \quad for\ 0 \le t \le \tau_1,$$

where τ_1 is defined as the first time when capital of vintage 1 is created and introduced into producing consumption goods. At $t = \tau_1$, we know that

$$q_0(\tau_1) = \frac{1}{a}\frac{\gamma - 1}{\gamma - 1/a}\frac{1}{bc(\tau_1)}.$$

It follows that $k_0(\tau_1) = [\frac{a-1}{a-1/\gamma}\frac{1}{\zeta}]^{\frac{\eta-1}{\eta}} = \ell(0)$. During this phase, $q_0(t) = q_0(0)e^{-(b-\rho)t}$, and $c(t) = k_0^1(t)$, leading to,

$$[k_0^1(t)^{-1} - \zeta k_0^1(t)^{\frac{1}{\eta-1}}] = [k_0^1(0)^{-1} - \zeta k_0^1(0)^{\frac{1}{\eta-1}}]e^{-(b-\rho)t}. \tag{1}$$

As $q_0(\tau_1) = q_0(0)e^{-(b-\rho)\tau_1}$, the length of the initial growth phase, τ_1, satisfies

$$\tau_1 = \frac{1}{b-\rho}\log\left\{a\frac{k_0^1(0)^{-1} - \zeta k_0^1(0)^{\frac{1}{\eta-1}}}{\frac{\gamma-1}{\gamma-1/a}\frac{1}{\ell(0)}}\right\}.$$

During this initial phase, the dynamics of $k_0(t)$ is

$$\dot{k}_0(t) = b[k_0(t) - k_0^1(t)],$$

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with k_0^1 given in Equation (4). This ODE does not admit analytical solutions, however, given an initial value $k_0^1(0)$, it uniquely determines the value of $k_0(t)$ for $0 < t \leq \tau_1$. At $t = \tau_1$, it must hold that $k_1(\tau_1)$ equals the steady state value of normalized capital stock calculated above. This equality uniquely pins down the initial value, $k_0^1(0)$.

Technology shocks in a CES production function Consider an aggregate CES production function, $y_t = e^{z_t}[\theta k_t^\rho + (1-\theta)\ell_t^\rho]^{\frac{1}{\rho}}$, with $\rho < 0$. A positive technology shock should immediately increase working hours. Capital, as a stock variable, increases much slower. Therefore, the capital-labor ratio shows a hump-shape after a positive technology shock. Consequently, the labor share displays an U-shape response in the labor share. Zheng (2007) has done a similar exercise. We show here that such a mechanism would produce variation in factor shares that is a too small under reasonable parameter values. To see this, note that wage is $w_t = e^{z_t}[\theta k_t^\rho + (1-\theta)\ell_t^\rho]^{\frac{1}{\rho}-1}(1-\theta)\ell^{\rho-1}$, and the labor share is $ls = \frac{w\ell}{y} = \frac{1-\theta}{1-\theta+\theta(\frac{k}{\ell})^\rho}$. Denote ls^* the steady state labor share, the standard deviation of ls satisfies

$$\sigma(ls) = \frac{\partial ls}{\partial \frac{k}{\ell}}(ls^{ss}) * \sigma(\frac{k}{\ell}) = \frac{1-\theta}{[(1-\theta)/ls^{ss}]^2}\theta\rho(\frac{1-\theta}{\theta})^{\frac{\rho-1}{\rho}}(\frac{1}{ls^{ss}}-1)^{\frac{\rho-1}{\rho}}\sigma(\frac{k}{\ell}) \equiv \Delta * \sigma(\frac{k}{\ell}).$$

Take the following parameter values, $ls^{ss} = 0.64$, $\theta = 0.3$ and $\rho = -0.25$ (to match an elasticity of substitution, $\frac{1}{1-\rho} = 0.8$), It follows that $\Delta = 0.019$. If $\rho = -0.5$, $\Delta = 0.067$. These values are too small to generate enough variations in $\sigma(ls)$ as observed in data. On the other hand, wage is $w = (1-\theta)(e^{z_t})^\rho(\frac{y}{\ell})^{1-\rho}$, which would generate a volatility of wage that is much larger than observed in data.

Appendix: Tables and Figures

Table 6.1: Fraction of *net operating surplus* in value added

	Initial Value	Max. Value	Increase	Final Value	Decrease
Expansion 60s	18.4%	23.6%	28.4%	18.0%	30.6%
Expansion 80s	15.4%	19.0%	23.5%	16.6%	14.5%
Expansion 90s	16.6%	19.9%	19.8%	15.8%	25.9%

Note: values are for in the non-financial corporate business sector. The first column is the fraction of the Net Operating Surplus at the beginning of the expansion, the second is the maximum value in the expansion, the third the percentage increase from initial to maximum; the fourth column is the final value at the end of the expansion, and the fifth the percentage decrease from peak to final.

Table 6.2: Moments in increasing *vs* decreasing KSTr periods

growth rate	Mean	Std. Dev.	Mean	Std. Dev.
	KSTr_IN		*KSTr_DE*	
value added	4.13	6.84	3.70	5.64
labor productivity	2.45	4.41	2.07	3.73
employment	1.68	3.93	1.93	3.22
working hours	1.62	4.74	1.60	3.84
real wage	0.88	3.49	2.02	3.22

Note: growth rate is percent change from previous quarter at annual rate for the non financial corporate sector from $1947q2$ to $2018q2$. KSTr_IN (KSTr_DE) denotes quarters when the HP trend of capital share increase (decrease). In total, there are 162 quarters with capital share going up, and 123 quarters with capital share going down. Quarters in the *KSTr_IN* subset contain: 1947q1-1950q3, 1958q4-1965q3, 1972q2-1984q3, 1991q4-1996q3, 2001q1-2013q2.

Figure 6.2: Gross capital share in the whole economy
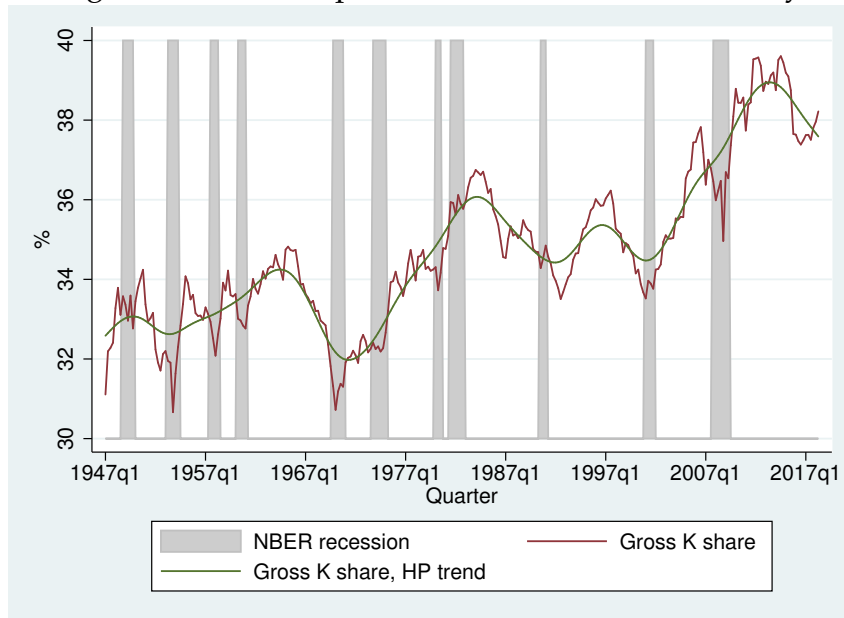
Figure 6.3: Gross capital share in the whole economy, non-depreciation components
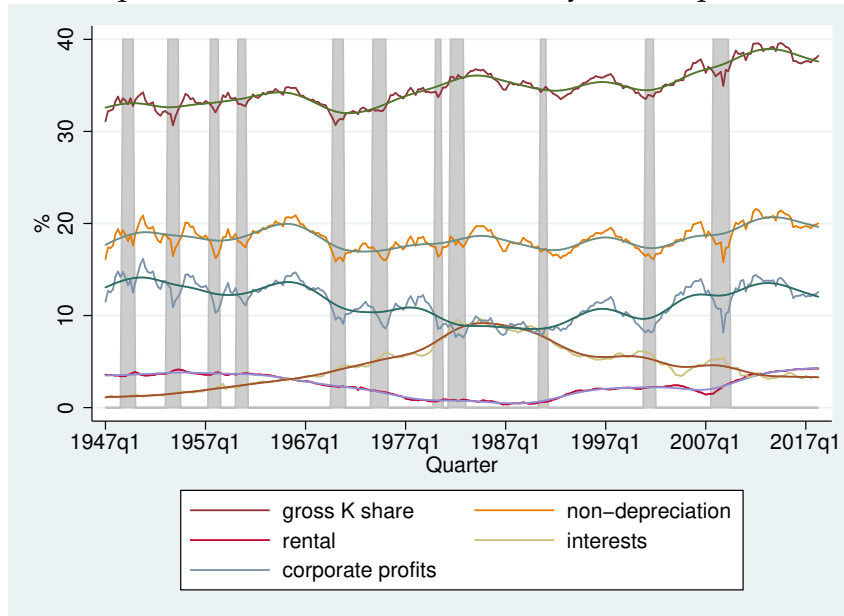
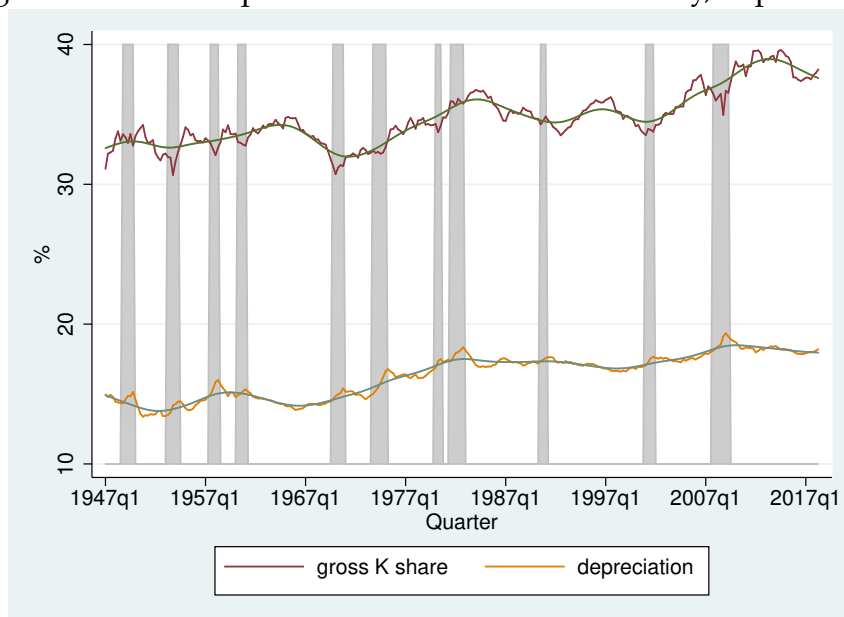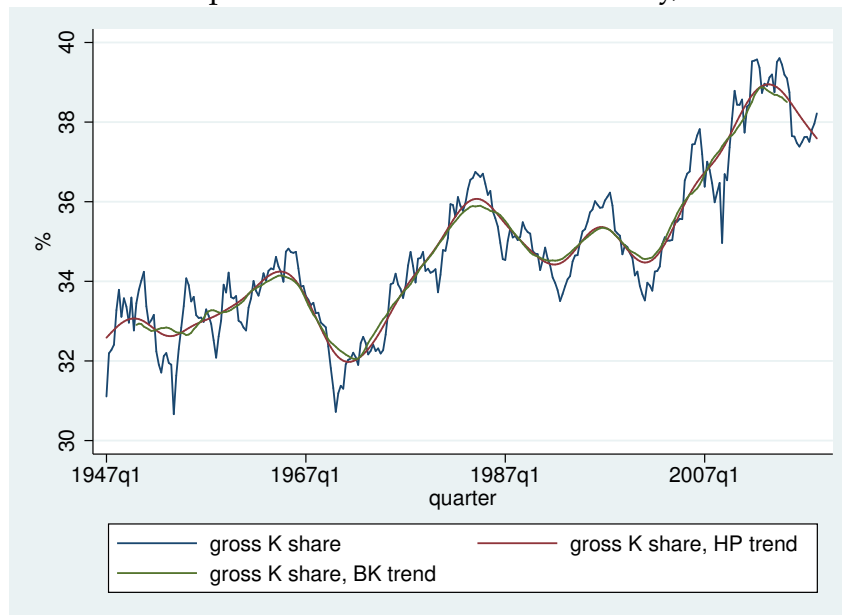Figure 6.4: Gross capital share in the whole economy, depreciation

Figure 6.5: Gross capital share in the whole economy, HP and BK trend

The smoothing parameter employed in HP filter is 1600, which is also what is used in the current paper. The second and additional trend is from the Baxter-King band pass filter with a lower limit frequency of 2 quarters and an upper limit frequency of 200 quarters (50 years). The choice of lower and upper limits in B-K filter follows Comin and Gertler (2006).

Figure 6.6: Net capital share in the whole economy
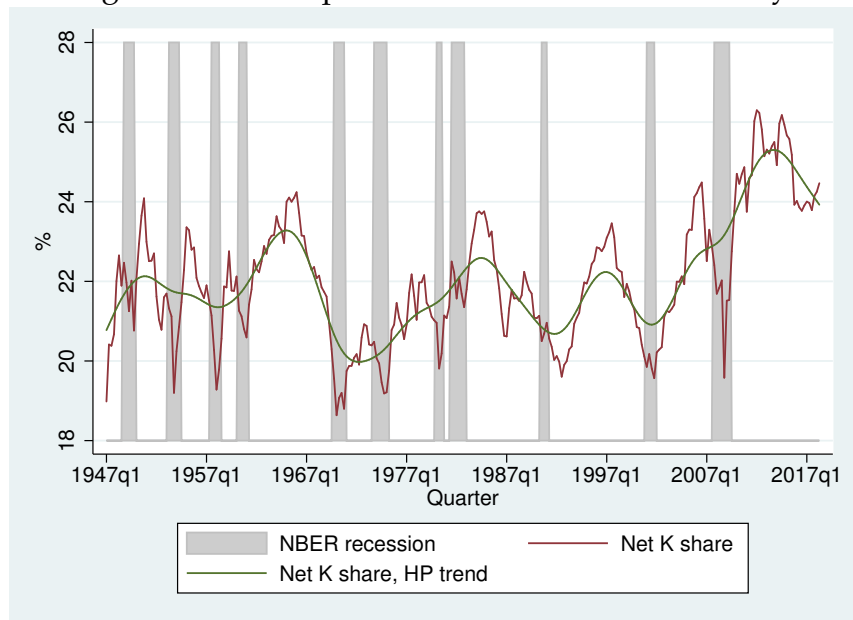
Figure 6.7: Gross capital share in the corporate business sector
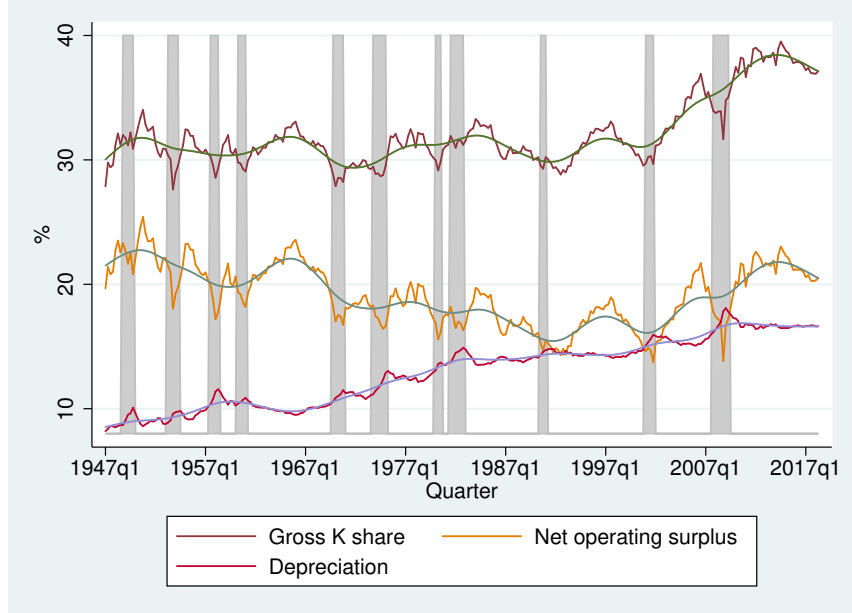
Figure 6.8: Net capital share in the corporate business sector
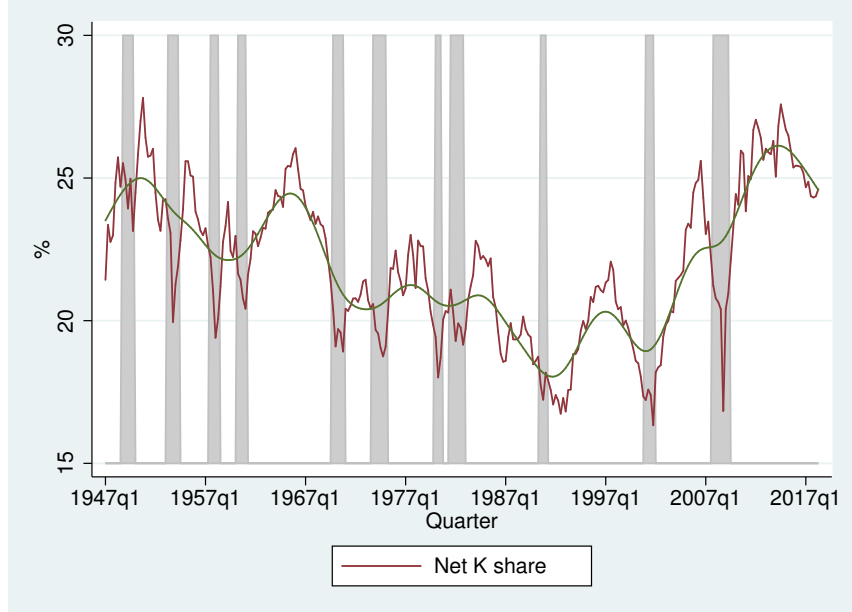
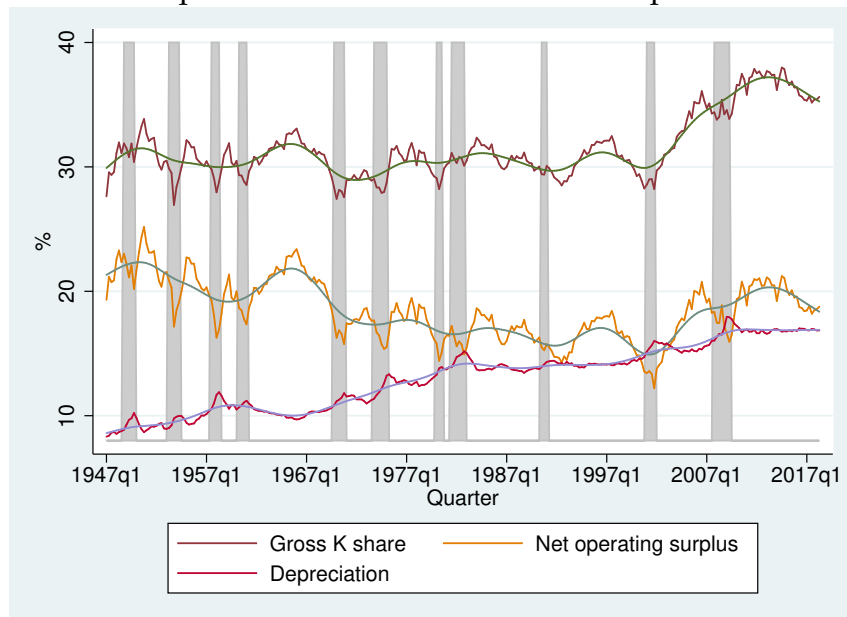Figure 6.9: Gross capital share in the nonfinancial corporate business sector

Figure 6.10: Net capital share in the nonfinancial corporate business sector
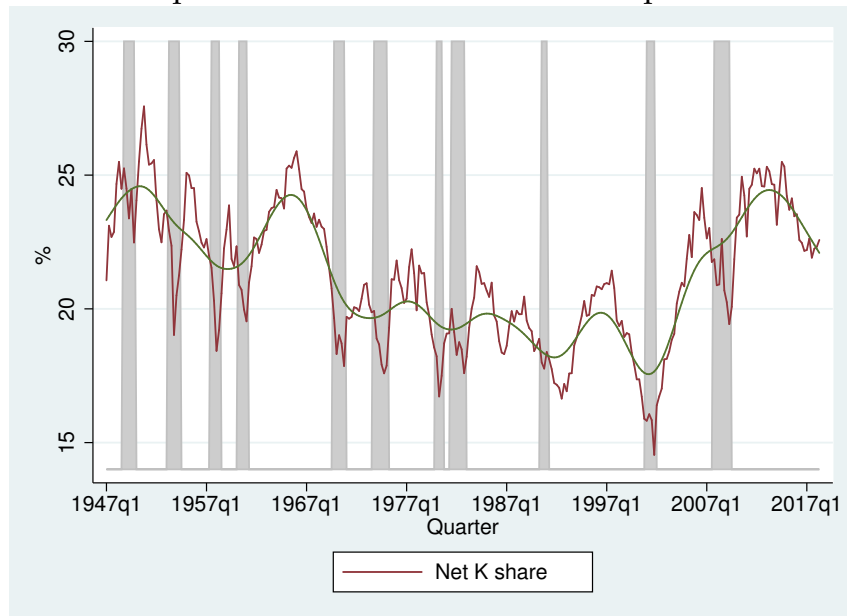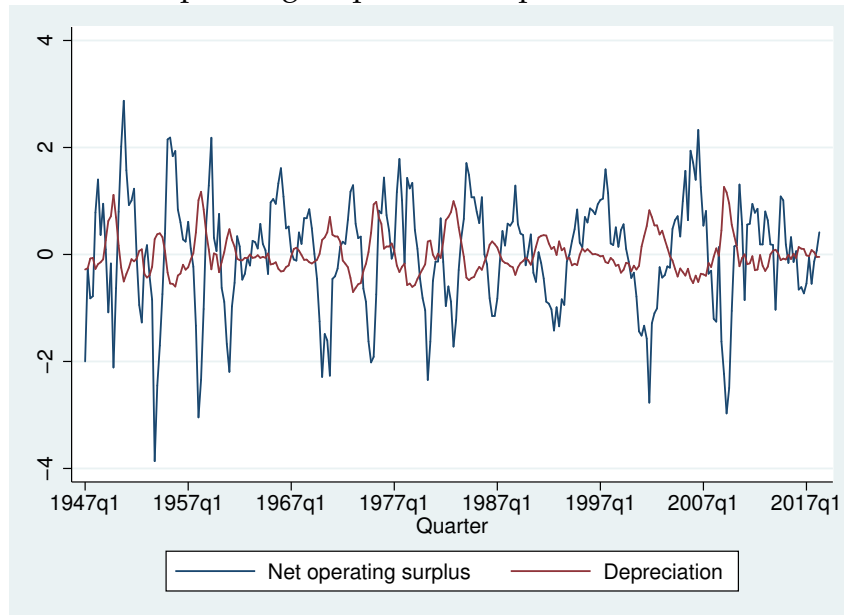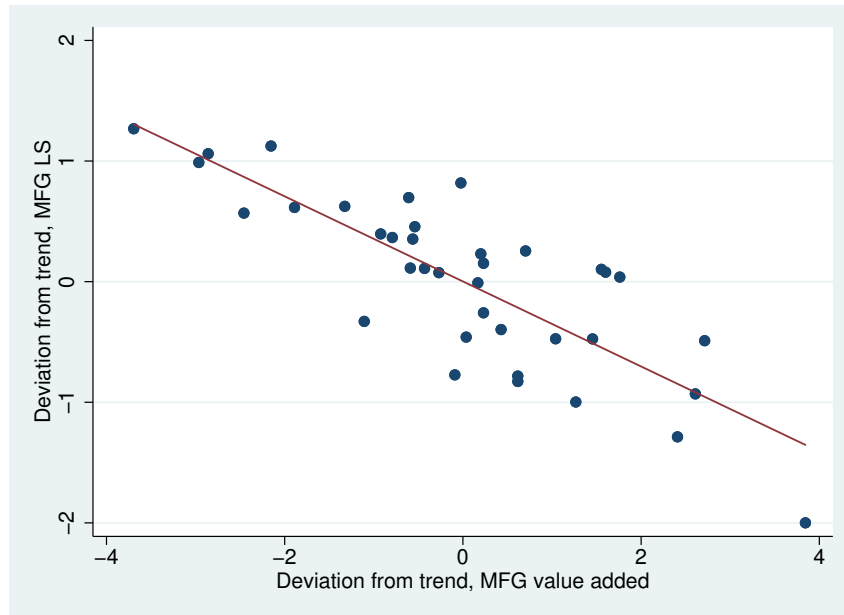
Figure 6.11: Net operating surplus and depreciation in the NFCB sector

Note: both series are HP detrended. NFCB-nonfinancial corporate business

Figure 6.12: Deviation from trend in LS *v.s.* Value added in the Manufacturing sector of France

Note: both series are HP de-trended.